1 Hyper-contraction of noise operators

In this section we begin the analysis of noise-correlation. The main interest here is understanding the correlation between $f(x_1, \ldots, x_n)$ and $f(y_1, \ldots, y_n)$ where $(x_1, \ldots, x_n)$ are chosen from a product distribution and $(y_1, \ldots, y_n)$ is obtained from $(x_1, \ldots, x_n)$ by applying some noise to each coordinate independently. The main difference in our study here compared to the study of influences will be our interest in re-randomizing many coordinates simultaneously, instead of studying the perturbation caused by a single parameter. Interestingly, our first application of this theory of noise-correlation will be to the study of influences.

We begin with a general definition of tensor product of operators – this corresponds to applying noise independently to each coordinate. Then we will study a strong property of these operators, named hyper-contraction – this will be used frequently later.

1.1 Noise operators

Definition 1 A operator $T : L^2(\mu) \to L^2(\mu)$ is called positivity improving if $Tf \geq 0$ for all $f \geq 0$. We will call $T$ a noise operator if it is positivity improving, $\|Tf\|_2 \leq \|f\|_2$ for all $f$, $T1 = 1$ and $\langle Tf, g \rangle = \langle f, Tg \rangle$ for all $f, g \in L^2(\mu)$.

Example 2 Let $(\Omega, \mu)$ be a finite probability space and let $M$ a Markov chain that is reversible with respect to $\mu$. $M$ corresponds to a non-negative $|M| \times |M|$ matrix that satisfies:

$$\sum_y M(x, y) = 1,$$

for all $x \in \Omega$ and

$$\mu(x)M(x, y) = \mu(y)M(y, x)$$

for all $x$ and $y$. Let $T_M$ be defined as follows

$$(T_Mf)(x) = \sum_y M(x, y)f(y).$$
Then $T_M$ is a noise operator:

$$
\|T_Mf\|_2^2 = \sum_x \mu(x)(T_Mf)^2(x) = \sum_{x,y} \mu(x)(M(x,y)f(y))^2
$$

$$
\leq \sum_{x,y} \mu(x)M(x,y)(f(y))^2 = \sum_y \mu(y)f^2(y) = \|f\|_2^2,
$$

$$(T_Mf,g) = \sum_x \mu(x)T_Mf(x)g(x) = \sum_{x,y} \mu(x)M(x,y)f(y)g(x) = \sum_{x,y} \mu(y)M(y,x)f(y)g(x) = (f,T_Mg).$$

**Example 3** Consider the space $L^2(\gamma_n)$ where $\gamma_n$ is the $n$-dimensional Gaussian measure. Let $0 \leq \rho \leq 1$. The Ornstein-Uhlenbeck operator is defined by:

$$
T_\rho f(x) = E_{y\sim\gamma_n}[f(\rho x + \sqrt{1-\rho^2}y)].
$$

In order to check that this is a noise operator note that

$$
E_{x\sim\gamma_n}[(T_\rho f)^2(x)] = E_{x\sim\gamma_n}\left[E_{y\sim\gamma_n}\left[f(\rho x + \sqrt{1-\rho^2}y)|x]\right]\right]
$$

$$
\leq E_{x\sim\gamma_n,y\sim\gamma_n}[f^2(\rho x + \sqrt{1-\rho^2}y)] = E_{x\sim\gamma_n}[f^2(x)],
$$

where the last equality follows from the fact that if $N_1, N_2$ are two independent standard Gaussian vectors, then so is $\rho N_1 + \sqrt{1-\rho^2}N_2$.

We also have that

$$
(T_\rho f,g) = E[f(X)f(Y)],
$$

where $(X,Y)$ is a normal $2n$-dimensional vector where $\text{Cov}[X_i, X_j] = \text{Cov}[Y_i, Y_j] = \delta_{i,j}$ and $\text{Cov}[X_i, Y_j] = \rho \delta_{i,j}$. Since this expression is symmetric in $X$ and $Y$ it follows that

$$
(T_\rho f,g) = (f,T_\rho g).
$$

### 1.2 Tensor products of noise operators

**Definition 4** Let $T_i : L^2(\mu_i) \to L^2(\mu_i)$ be a bounded linear operator. Let $U^i$ be a basis of $L^2(\mu_i)$. We define $T = \otimes_{i=1}^n T_i$ to be the linear operator satisfying

$$
T(\otimes_{i=1}^n u_i) = \otimes_{i=1}^n (T_i u_i),
$$

for every basis element $\otimes_{i=1}^n u_i$.

This definition roughly says that $T$ acts on coordinates $i$ by $T_i$. One needs to check that this definition does not depend on the choice of basis.
Lemma 5 The operator $T$ does not depend on the choice of basis.

Proof: We need to prove that for any two bases $\otimes_{i=1}^n U^i$ and $\otimes_{i=1}^n V^i$ we get the same operator. Clearly it suffices to show that assuming $U^i = V^i$ except at a single coordinate $i$ that we may assume WLOG is 1. In other words, it suffices to show we obtain the same operator for $U^1 \otimes \ldots \otimes U^n$ and for $V^1 \otimes U^2 \otimes U^n$. This follows immediately from the linearity of $T_i$. □

Lemma 6 Let $T_i : L^2(\mu_i) \to L^2(\mu_i)$ be a bounded linear operators. Let $T_i^* : L^2(\prod_{i=1}^n \mu_i) \to L^2(\prod_{i=1}^n \mu_i)$ be defined by

$$(T_i^* f(x_1, \ldots, x_n))(x_i) = (T_i f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n))(x_i).$$

Then $\prod_{i=1}^n T_i^* = \otimes_{i=1}^n T_i$ and the operators $T_i^*$ commute.

Proof: It suffices to check that $\prod_{i=1}^n T_i^* = \prod_{i=1}^n T_i$ for basis elements. □

Lemma 7 If $T_1, \ldots, T_n$ are noise operators then so is $\otimes_{i=1}^n T_i$.

Proof: It is easy to see that each of the $T_i^*$ is a noise operator. □

Lemma 8 Suppose $T^i$ is a Markov operator on $L^2(\mu_i)$ that is defined by a reversible Markov chain $M^i$. Then the operator $\otimes_{i=1}^n T^i$ is the operators defined by the Markov chain $M$ where,

$$M(x, y) = \prod_{i=1}^n M^i(x_i, y_i).$$

Proof: It suffices to show that the two operators acts the same on tensors. Let $u = \otimes_{i=1}^n u_i$ be such a tensor then

$$(T_M u)(x) = \sum_y M(x, y) u(y) = \sum_y \prod_{i=1}^n M^i(x_i, y_i) u_i(y_i) = \prod_{i=1}^n \left( \sum_{y_i} M^i(x_i, y_i) u_i(y_i) \right) = \prod_{i=1}^n T^i(u_i),$$

as needed. □

Example 9 The most important noise operator we will study is the Bonami-Beckner operator. This operators is specified by a single paramter $0 \leq \rho \leq 1$. The operator $T^\rho$ is defined on $L^2(\prod_{i=1}^n \mu_i)$ by $T^\rho = \otimes_{i=1}^n T_i^\rho$, where $T_i^\rho(f) = \rho f + (1 - \rho)E[f]$. Note that the operator $T_i^\rho$ may be defined via the Markov chain $M^i$ where $M^i(x, y) = \rho \delta_x + (1 - \rho)\mu(y)$. Therefore the operator $T^\rho$ corresponds to $M(x, y)$ where $y_i = x_i$ with probability $\rho$ and is chosen independently from the measure $\mu$ independently for all $i$. 

11-3
Noise operators are contractions by definition. They satisfy $\|Tf\|_2 \leq \|f\|_2$. More importantly, many of these operators are hyper-contractive.

**Definition 10** Let $1 \leq p \leq q$ then we say that the operator $T$ is $(p,q)$-hypercontractive satisfies $\|Tf\|_q \leq \|f\|_p$ for every $f$ with $\|f\|_p < \infty$. 