Lecture 0

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Consider finite probability spaces $\Omega_1, \ldots, \Omega_n$, with measures $\mu_1, \ldots, \mu_n$. Let α_i be size of the smallest atom of (Ω_i, μ_i), and set $\alpha = \min_i \alpha_i$. Let $f \in L^2(\prod_i \mu_i)$ be a real function. Let $\Delta_i f = \sum_{S:i\in S} \hat{f}(S) U_S$.

Theorem 1 (Generalizaion of Talagrand, 1994) *There exists some universal constant C such that*

$$var(f) \leq C \log(1/\alpha) \sum_{i\leq n} \frac{||\Delta_i f||_2^2}{\log\Big(||\Delta_i f||_2/||\Delta_i f||_1\Big)}\,.$$

Corollary 2 (Kahn, Kalai and Linial, 1988) *Consider $f : \{0,1\}^n \to \{0,1\}$, where $\{0,1\}^n$ is endowed with the uniform measure, then there exists a constant $C > 0$ such that*

$$\max_i I_i(f) \geq C var(f) \frac{\log n}{n}\,.$$

Proof: [of Corollary 2] Recall that $||\Delta_i f||_2^2 = I_i(f)$, and that $x/\log(1/x)$ is increasing on $(0,1)$. By the identity $\Delta_i f = f - E[f \mid X_j, \quad j \neq i]$, it is easy to check that that $||\Delta_i f||_1 = I_i(f)$. So by Theorem 1 we get

$$C var(f) \leq n \frac{\max_i I_i(f)}{\log(\max_i I_i(f))}\,,$$

and since $y/\log(1/y) \geq x$ implies $y \geq Kx/\log(1/x)$ for some constant K for all $x \in (0,1/2)$, we get the result. □

Remark 3 *Similarly we can prove that for all $p \in (0,1)$ there exists a constant C_p such that if $f : \{0,1\}^n \to \{0,1\}$, where $\{0,1\}^n$ is endowed with the $Bin(n,p)$ measure, then*

$$\max_i I_i(f) \geq C_p var(f) \frac{\log n}{n}\,.$$

Proof: [of Theorem 1] For a real function g from our space, denote

$$M^2(g) = \sum_{S:i\in S} \frac{\hat{g}(S)^2}{|S|}\,.$$

So

$$var(f) = \sum_{S\neq\emptyset} \hat{f}(S)^2 = \sum_{i\leq n} M^2(\Delta_i f)\,,$$

and hence it suffices to prove that for any function g with $\mathbf{E}g = 0$,

$$M^2(g) \leq K\log(1/\alpha)\frac{||g||_2^2}{\log\Big(||g||_2/||g||_1\Big)}\,. \qquad (1)$$

To prove (1) we use hypercontractivity. The following proposition is proved in the end of this note.

Proposition 4 *Let $q \in (1,2)$ and $\Theta \in (0,1)$ satisfies*

$$\Theta^2 \leq \frac{\alpha^2}{3}(q-1)\,,$$

then for all functions g we have,

$$||T_\Theta g||_2 \leq ||g||_q\,,$$

where T_Θ is the Bonami-Beckner operator.

Recall that

$$T_\Theta g = \sum_S \Theta^{|S|}\hat{g}(S)U_S\,,$$

and apply the previous with $q = 3/2$, and $\Theta^2 = \frac{\alpha^2}{6}$. This gives that for any integer $k > 0$,

$$\Theta^{2k}\sum_{|S|=k}\hat{g}(S)^2 \leq \sum_S \Theta^{2|S|}\hat{g}(S)^2 = ||T_\Theta g||_2^2 \leq ||g||_{3/2}^2\,,$$

hence

$$\sum_{|S|=k}\hat{g}(S)^2 \leq \Big(\frac{6}{\alpha^2}\Big)^k ||g||_{3/2}^2\,.$$

Fix an integer $m > 0$, and sum the previous for all $k \leq m$ to get

$$\sum_{|S|\leq m}\frac{\hat{g}(S)^2}{|S|} \leq \sum_{k\leq m}\frac{\Big(\frac{6}{\alpha^2}\Big)^k}{k}||g||_{3/2}^2 \leq \frac{2\Big(\frac{6}{\alpha^2}\Big)^m}{m}||g||_{3/2}^2\,,$$

where the last inequality comes from the fact that the ratio between two consecutive summands in the sum is greater than 2. We now have

$$\begin{aligned} M^2(g) &= \sum_{|S|\leq m}\frac{\hat{g}(S)^2}{|S|} + \sum_{|S|>m}\frac{\hat{g}(S)^2}{|S|} \leq \frac{2\Big(\frac{6}{\alpha^2}\Big)^m}{m}||g||_{3/2}^2 + \frac{||g||_2^2}{m} \\ &\leq \frac{2}{m}\Big[\Big(\frac{6}{\alpha^2}\Big)^m ||g||_{3/2}^2 + ||g||_2^2\Big]\,. \qquad (2) \end{aligned}$$

We now choose optimal m. Choose largest m such that $\left(\frac{6}{\alpha^2}\right)^m ||g||_{3/2}^2 \leq ||g||_2^2$, hence

$$\left(\frac{6}{\alpha^2}\right)^{m+1} ||g||_{3/2}^2 \geq ||g||_2^2 \implies m+1 \geq \frac{2\log\Big(||g||_2/||g||_{3/2}\Big)}{\log(6/\alpha^2)}\,.$$

Plugging this back into (2) gives

$$M^2(g) \leq C\frac{\log(6/\alpha^2)||g||_2^2}{\log\Big(||g||_2/||g||_{3/2}\Big)}\,.$$

An application of Cauchy-Schwartz gives

$$||g||_{3/2}^3 \leq ||g||_1 ||g||_2^2\,,$$

hence

$$\left(\frac{||g||_{3/2}}{||g||_2}\right)^3 \leq \frac{||g||_1}{||g||_2}\,,$$

which concludes the proof of (1) and so we are done. □

Let $A \subset \{0,1\}^n$ be a monotone increasing set. Let μ_p be the $\mathrm{Bin}(n,p)$ measure on $\{0,1\}^n$. Note that since A is increasing, $\mu_p(A)$ is an increasing function in p. Moreover, it is a polynomial and in particular it is infinitely differentiable.

Lemma 5 (Russo's Lemma)

$$\frac{\partial \mu_p(A)}{\partial p} = \frac{\sum_{i\leq n} I_i^{(p)}(A)}{p(1-p)}\,.$$

Proof: Let $\varphi(p_1, p_2, \ldots, p_n) : [0,1]^n \to [0,1]$ be a function returning the measure of A in the space $L^2\Big(\prod_i \mu_i\Big)$ where μ_i is a measure on the two point space $\{0,1\}$ which gives 1 weight p_i and gives 0 weight $1-p_i$. The clearly $\mu_p(A) = \varphi(p, \ldots, p)$, so by the chain rule

$$\frac{\partial \mu_p(A)}{\partial p} = \sum_{i\leq n} \frac{\partial \varphi}{\partial p_i}(p, \ldots, p) = \sum_{i\leq n} \frac{I_i^{(p)}(A)}{p(1-p)}\,,$$

where the last equality is due to the easy fact

$$\frac{\partial \varphi}{\partial p_i}(p, \ldots, p) = \frac{I_i^{(p)}(A)}{p(1-p)}\,.$$

□

A graph property P on n vertices is a set of graphs on n vertices which is invariant under vertex permutations. The following theorem states that any graph property which is monotone experiences a 'sharp threshold'.

Theorem 6 (Friedgut and Kalai, 1996) *Let P be a monotone increasing graph property on n vertices. If $p \in (0,1)$ is such that $\mu_p(P) > \epsilon$, then*

$$\mu_q(P) > 1 - \epsilon\,,$$

for $q = p + c_1 \frac{\log(\frac{1}{2\epsilon})}{\log n}$, where $c_1 > 0$ is a universal constant.

Proof: Invariance under vertex permutation gives that all influences of the indicator function of A are equal (note the edges of graph are the variables of the function). Hence by Theorem 1 and Remark 3 we have that

$$\sum_i I_i(A) \geq C\mu_p(A)(1-\mu_p(A))\log n\,.$$

For any $r > p$ such that $\mu_r(A) \leq 1/2$, by Lemma 5 and the previous line we have that

$$\frac{\partial \mu_r(A)}{\partial r} \geq C\mu_r(A)\log n\,,$$

where we consider p to be fixed (and hence so is $1/p$). Last equation can be written as

$$\frac{\partial \log(\mu_r(A))}{\partial r} \geq C\log n\,,$$

and so if we take $q' = p + \frac{\log(\frac{1}{2\epsilon})}{C\log n}$ we get by the fundamental theorem of calculus that

$$\log(\mu_{q'}(A)) \geq \log(\mu_p(A)) + \int_p^{q'} C\log n \geq \log(\epsilon) + \log(\frac{1}{2\epsilon}) = \log(1/2)\,.$$

And so $\mu_{q'}(A) \geq 1/2$. Similarly, if we take $q = q' + \frac{\log(\frac{1}{2\epsilon})}{C\log n}$ we get that $\mu_q(A) \geq 1-\epsilon$.

□

Proof: [of Proposition 4] We have learned that the hypercontractive constant for the space $L^2\Big(\prod_i \mu_i\Big)$ is

$$\Theta(q) = \Big(\frac{(1-\alpha)^{2-2/q} - \alpha^{2-2/q}}{(1-\alpha)\alpha^{1-2/q} - \alpha(1-\alpha)^{1-2/q}}\Big)^{1/2},$$

for all $q \in (1,2)$. Thus in order to prove the claim, we just need to lower bound $\Theta(q)$. Let

$$f(x) = x^{2-2/q}, \quad g(x) = -(1-x)x^{1-2/q}\,,$$

and by Lagrange's theorem we have

$$\Theta(q)^2 = \frac{f'(\xi_1)}{g'(\xi_2)}\,,$$

for some $\xi_1, \xi_2 \in (\alpha, 1-\alpha)$. By computing, one can check that f' and g' are decreasing, and hence

$$\begin{aligned}
\Theta(q)^2 &\geq \frac{f'(1-\alpha)}{g'(\alpha)} = \frac{(2-2/q)(1-\alpha)^{1-2/q}}{\alpha^{1-2/q} + (2/q-1)\alpha^{-2/q}(1-\alpha)} \\
&= \frac{2(q-1)}{q}\Big(\frac{1-\alpha}{\alpha}\Big)^{-2/q}\Big[\frac{1-\alpha}{\alpha + (2/q-1)(1-\alpha)}\Big] \geq \frac{(q-1)\alpha^2}{3}.
\end{aligned}$$

□