Consider finite probability spaces $\Omega_1, \ldots, \Omega_n$, with measures $\mu_1, \ldots, \mu_n$. Let $\alpha_i$ be the size of the smallest atom of $(\Omega_i, \mu_i)$, and set $\alpha = \min_i \alpha_i$. Let $f \in L^2(\prod_i \mu_i)$ be a real function. Let $\Delta_i f = \sum_{S: i \in S} \hat{f}(S) \mathbf{1}_S$.

**Theorem 1 (Generalization of Talagrand, 1994)** There exists some universal constant $C$ such that

$$\text{var}(f) \leq C \log(1/\alpha) \sum_{i \leq n} \frac{\|\Delta_i f\|^2_2}{\log \left( \|\Delta_i f\|_2 / \|\Delta_i f\|_1 \right)}.$$ 

**Corollary 2 (Kahn, Kalai and Linial, 1988)** Consider $f : \{0,1\}^n \to \{0,1\}$, where $\{0,1\}^n$ is endowed with the uniform measure, then there exists a constant $C > 0$ such that

$$\max_i I_i(f) \geq C \text{var}(f) \frac{\log n}{n}.$$ 

**Proof:** [of Corollary 2] Recall that $\|\Delta_i f\|^2_2 = I_i(f)$, and that $x/\log(1/x)$ is increasing on $(0,1)$. By the identity $\Delta_i f = f - E[f \mid X_j, \ j \neq i]$, it is easy to check that that $\|\Delta_i f\|_1 = I_i(f)$. So by Theorem 1 we get

$$C \text{var}(f) \leq n \frac{\max_i I_i(f)}{\log(\max_i I_i(f))},$$

and since $y/\log(1/y) \geq x$ implies $y \geq K x / \log(1/x)$ for some constant $K$ for all $x \in (0,1/2)$, we get the result. □

**Remark 3** Similarly we can prove that for all $p \in (0,1)$ there exists a constant $C_p$ such that if $f : \{0,1\}^n \to \{0,1\}$, where $\{0,1\}^n$ is endowed with the Bin($n, p$) measure, then

$$\max_i I_i(f) \geq C_p \text{var}(f) \frac{\log n}{n}.$$ 

**Proof:** [of Theorem 1] For a real function $g$ from our space, denote

$$M^2(g) = \sum_{S: x \in S} \frac{\hat{g}(S)^2}{|S|}.$$
So
\[ \text{var}(f) = \sum_{S \neq \emptyset} \hat{f}(S)^2 = \sum_{i \leq n} M^2(\Delta_i f), \]
and hence it suffices to prove that for any function \( g \) with \( \mathbf{E}g = 0 \),
\[ M^2(g) \leq K \log(1/\alpha) \frac{||g||^2_2}{\log \left( ||g||_2/||g||_1 \right)}. \]  
(1)

To prove (1) we use hypercontractivity. The following proposition is proved in the end of this note.

**Proposition 4** Let \( q \in (1, 2) \) and \( \Theta \in (0, 1) \) satisfies
\[ \Theta^2 \leq \frac{\alpha^2}{3} (q - 1), \]
them for all functions \( g \) we have,
\[ ||T_\Theta g||_2 \leq ||g||_q, \]
where \( T_\Theta \) is the Bonami-Beckner operator.

Recall that
\[ T_\Theta g = \sum_S \Theta^{|S|} \hat{g}(S) U_S, \]
and apply the previous with \( q = 3/2 \), and \( \Theta^2 = \frac{\alpha^2}{6} \). This gives that for any integer \( k > 0 \),
\[ \Theta^{2k} \sum_{|S|=k} \hat{g}(S)^2 \leq \sum_S \Theta^{2|S|} \hat{g}(S)^2 = ||T_\Theta g||^2_2 \leq ||g||^2_{3/2}, \]
hence
\[ \sum_{|S|=k} \hat{g}(S)^2 \leq \left( \frac{6}{\alpha^2} \right)^k ||g||^2_{3/2}. \]

Fix an integer \( m > 0 \), and sum the previous for all \( k \leq m \) to get
\[ \sum_{|S| \leq m} \frac{\hat{g}(S)^2}{|S|} \leq \sum_{k \leq m} \left( \frac{6}{\alpha^2} \right)^k \frac{||g||^2_{3/2}}{k} \leq \frac{2}{m} \left( \frac{6}{\alpha^2} \right)^m ||g||^2_{3/2}, \]
where the last inequality comes from the fact that the ratio between two consecutive summands in the sum is greater than 2. We now have
\[ M^2(g) = \sum_{|S| \leq m} \frac{\hat{g}(S)^2}{|S|} + \sum_{|S| > m} \frac{\hat{g}(S)^2}{|S|} \leq \frac{2}{m} \left( \frac{6}{\alpha^2} \right)^m ||g||^2_{3/2} + \frac{||g||^2_2}{m}, \]
(2)
We now choose optimal $m$. Choose largest $m$ such that \[
\left( \frac{6}{\alpha^2} \right)^m \|g\|_{3/2}^2 \leq \|g\|_2^2, \]
hence
\[
\left( \frac{6}{\alpha^2} \right)^{m+1} \|g\|_{3/2}^2 \geq \|g\|_2^2 \implies m + 1 \geq \frac{2 \log \left( \frac{\|g\|_2}{\|g\|_{3/2}} \right)}{\log (6/\alpha^2)}. \]
Plugging this back into (2) gives
\[
M^2(g) \leq C \frac{\log (6/\alpha^2) \|g\|_2^2}{\log \left( \frac{\|g\|_2}{\|g\|_{3/2}} \right)}. \]
An application of Cauchy-Schwartz gives
\[
\|g\|_2^3 \leq \|g\|_1 \|g\|_2^2, \]
hence
\[
\left( \frac{\|g\|_2^3}{\|g\|_2^2} \right)^3 \leq \frac{\|g\|_1}{\|g\|_2}, \]
which concludes the proof of (1) and so we are done. □

Let $A \subset \{0, 1\}^n$ be a monotone increasing set. Let $\mu_p$ be the Bin$(n, p)$ measure on $\{0, 1\}^n$. Note that since $A$ is increasing, $\mu_p(A)$ is an increasing function in $p$. Moreover, it is a polynomial and in particular it is infinitely differentiable.

**Lemma 5 (Russo’s Lemma)**
\[
\frac{\partial \mu_p(A)}{\partial p} = \frac{\sum_{i \leq n} I_i^{(p)}(A)}{p(1-p)}. \]

**Proof:** Let $\varphi(p_1, p_2, \ldots, p_n) : [0, 1]^n \to [0, 1]$ be a function returning the measure of $A$ in the space $L^2(\prod_i \mu_i)$ where $\mu_i$ is a measure on the two point space $\{0, 1\}$ which gives 1 weight $p_i$ and gives 0 weight $1 - p_i$. The clearly $\mu_p(A) = \varphi(p, \ldots, p)$, so by the chain rule
\[
\frac{\partial \mu_p(A)}{\partial p} = \sum_{i \leq n} \frac{\partial \varphi}{\partial p_i}(p_1, \ldots, p) = \sum_{i \leq n} \frac{I_i^{(p)}(A)}{p(1-p)}, \]
where the last equality is due to the easy fact
\[
\frac{\partial \varphi}{\partial p_i}(p_1, \ldots, p) = \frac{I_i^{(p)}(A)}{p(1-p)}. \]
□

A graph property $P$ on $n$ vertices is a set of graphs on $n$ vertices which is invariant under vertex permutations. The following theorem states that any graph property which is monotone experiences a 'sharp threshold'.

0-3
Theorem 6 (Friedgut and Kalai, 1996) Let $P$ be a monotone increasing graph property on $n$ vertices. If $p \in (0,1)$ is such that $\mu_p(P) > \epsilon$, then

$$\mu_q(P) > 1 - \epsilon,$$

for $q = p + c_1 \frac{\log(\frac{1}{2})}{\log n}$, where $c_1 > 0$ is a universal constant.

**Proof:** Invariance under vertex permutation gives that all influences of the indicator function of $A$ are equal (note the edges of graph are the variables of the function). Hence by Theorem 1 and Remark 3 we have that

$$\sum_i I_i(A) \geq C \mu_p(A)(1 - \mu_p(A)) \log n.$$ 

For any $r > p$ such that $\mu_r(A) \leq 1/2$, by Lemma 5 and the previous line we have that

$$\frac{\partial \mu_r(A)}{\partial r} \geq C \mu_r(A) \log n,$$

where we consider $p$ to be fixed (and hence so is $1/p$). Last equation can be written as

$$\frac{\partial \log(\mu_r(A))}{\partial r} \geq C \log n,$$

and so if we take $q' = p + \frac{\log(\frac{1}{2})}{C \log n}$ we get by the fundamental theorem of calculus that

$$\log(\mu_{q'}(A)) \geq \log(\mu_p(A)) + \int_p^{q'} C \log n \geq \log(\epsilon) + \log(\frac{1}{2\epsilon}) = \log(1/2).$$

And so $\mu_{q'}(A) \geq 1/2$. Similarly, if we take $q = q' + \frac{\log(\frac{1}{2})}{C \log n}$ we get that $\mu_q(A) \geq 1 - \epsilon$.

$\square$

**Proof:** [of Proposition 4] We have learned that the hypercontractive constant for the space $L^2(\prod \mu_i)$ is

$$\Theta(q) = \left( \frac{(1 - \alpha)^{2-2/q} - \alpha^{2-2/q}}{(1 - \alpha)\alpha^{1-2/q} - \alpha(1 - \alpha)^{1-2/q}} \right)^{1/2},$$

for all $q \in (1, 2)$. Thus in order to prove the claim, we just need to lower bound $\Theta(q)$. Let

$$f(x) = x^{2-2/q}, \quad g(x) = -(1 - x)x^{1-2/q},$$

and by Lagrange’s theorem we have

$$\Theta(q)^2 = \frac{f''(\xi_1)}{g'(\xi_2)},$$
for some $\xi_1, \xi_2 \in (\alpha, 1 - \alpha)$. By computing, one can check that $f'$ and $g'$ are decreasing, and hence

$$\Theta(q)^2 \geq \frac{f'(1 - \alpha)}{g'(\alpha)} = \frac{(2 - 2/q)(1 - \alpha)^{1-2/q}}{\alpha^{1-2/q} + (2/q - 1)\alpha^{-2/q}(1 - \alpha)}$$

$$= \frac{2(q - 1)}{q} \left(\frac{1 - \alpha}{\alpha}\right)^{-2/q} \left[\frac{1 - \alpha}{\alpha + (2/q - 1)(1 - \alpha)}\right] \geq \frac{(q - 1)\alpha^2}{3}.$$