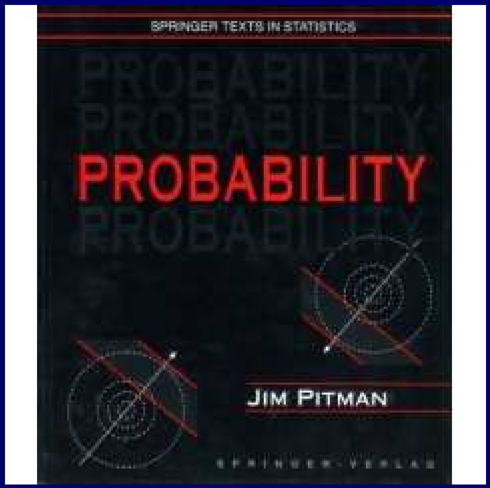
Introduction to probability

Stat 134 FAll 2005

Berkeley



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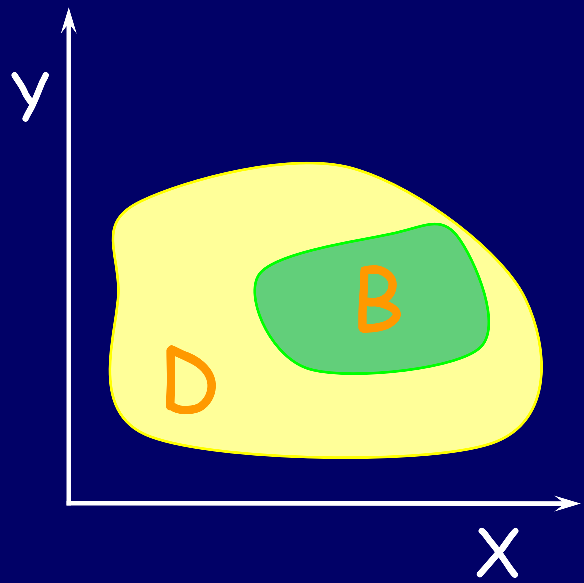
Follows Jim Pitman's book:

Probability

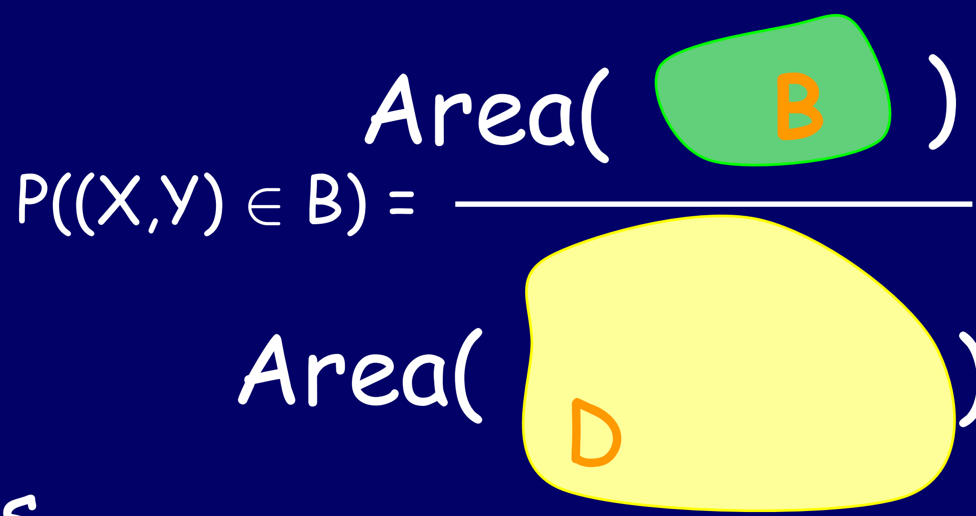
Section 5.1

Uniform distribution in an area



- Sample space is D.
- Outcomes are points in D with random coordinates (X,Y).

A



- Probability of each subset is equal to its relative area.

Joint Distributions

Suppose that $X \sim Unif(0,a)$, $Y \sim Unif(0,b)$ and X and Y are independent.

Claim: Then the random pair (X,Y) is uniformly distributed in the rectangle $([0,a] \times [0,b])$.

Proof:

$P((X,Y) \in A \times B)) = P(X \in A \ \& \ Y \in B)$

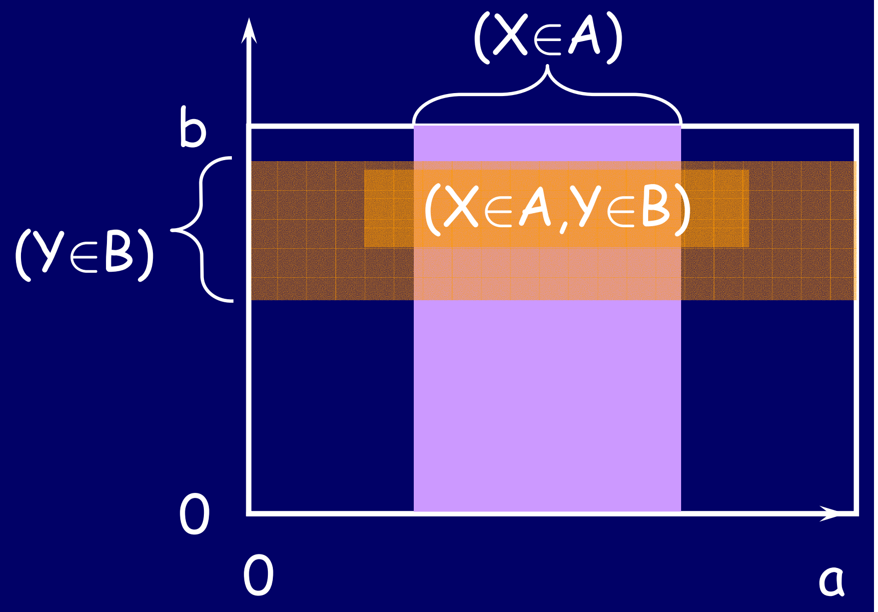
$= P(X \in A)\, P(Y \in B)$ (by ind.)

$= \text{length}(A)/a \times \text{length}(B)/b$

$= \text{area}(A \times B)/(ab)$.

$\Rightarrow P((X,Y) \in C) = Area(C)/ab$ for finite union of rectangles.

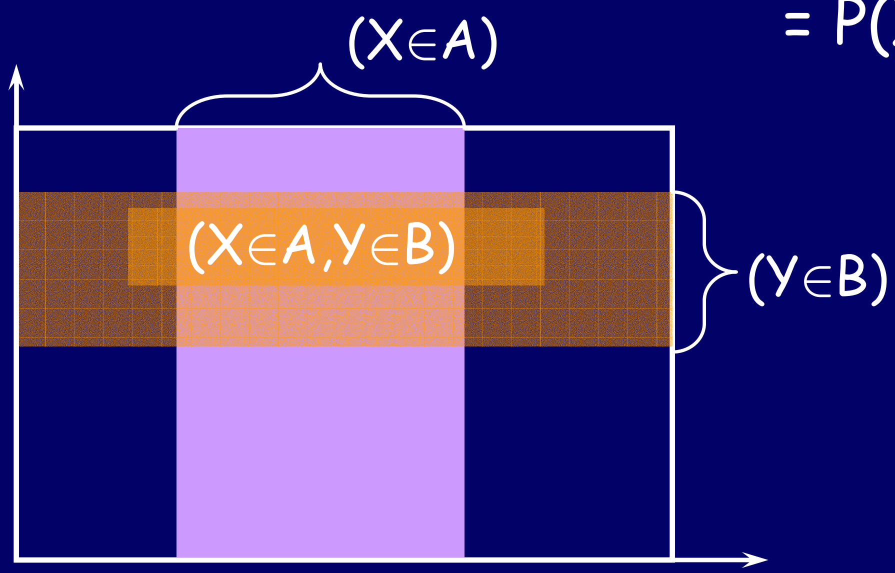
$\Rightarrow P((X,Y) \in C) = Area(C)/ab$ for all C.



Joint Distributions

Claim: Conversely if $(X,Y) \sim \text{Unif}([0,a] \times [0,b])$ then X and Y are independent.

Proof: $P(X \in A \ \&\ Y \in B)$ $= P((X,Y) \in A \times B)) = \text{area}(A \times B)/(ab)$
$= \text{length}(A)/a \times \text{length}(B)/b$
$= P(X \in A)\, P(Y \in B)$



Question: If you know X=x, what can you say about the distribution of

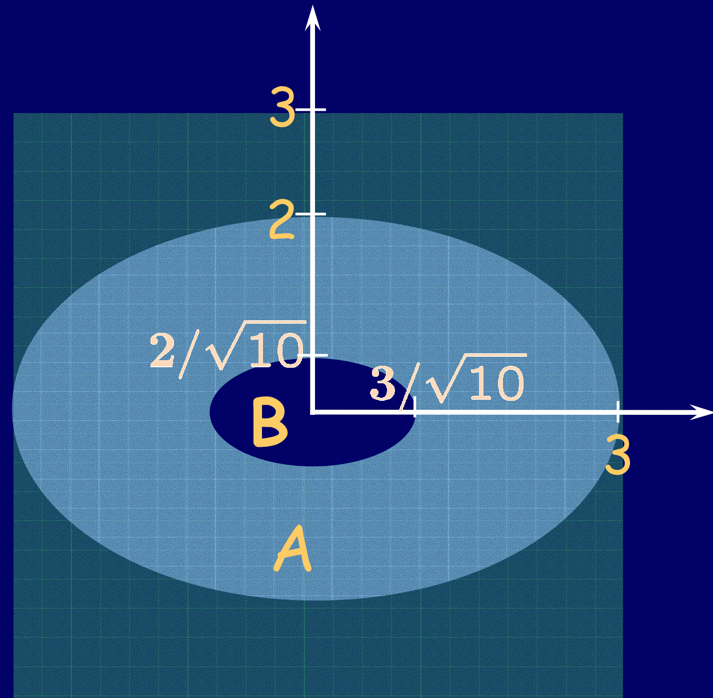
Y? X+Y? X-Y? X*Y?

Question: Is the rectangle the only shape for which X & Y are independent?

Joint Distributions

Example: $(X,Y) \sim \text{Unif}([-3,3]\times[-3,3])$

Find $P(4X^2 + 9Y^2 \leq 36 \;\;\&\;\; 40X^2 + 90Y^2 \geq 36)$.



Area(A+B)= $\pi*3*2 = 6\pi$.

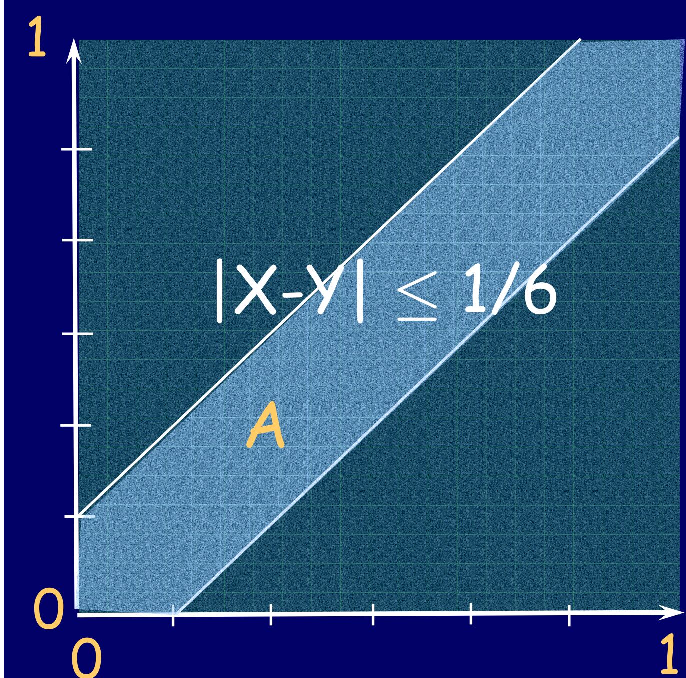
Area(B) = 0.6π.

Area(A)= 5.4π.

$P(A) = 5.4\pi / 36$

Rendezvous at a Coffee Shop

Question: A boy and a girl frequent the same coffee shop. Each arrives at a random time between 5 and 6 pm, independently of the other, and stays for 10 minutes. What's the chance that they would meet?



Solution: Let

X = her arrival time

Y = his arrival time.

Then $(X,Y) \sim \text{Unif}([0,1] \times [0,1])$.

They meet if $|X-Y| \leq 1/6$.

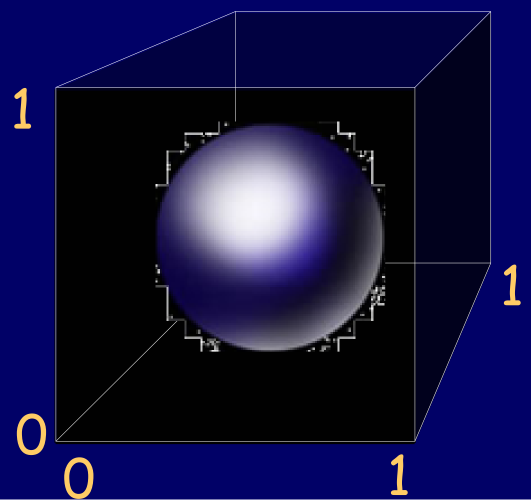
$P[|X-Y| \leq 1/6] = 1 - (5/6)^2$

Uniform Distribution over Volume

Claim: If $U_1, U_2, \ldots, U_n$ are independent variables such that $U_i \sim \mathrm{Unif}(0,1)$, then $(U_1, U_2, \ldots, U_n) \sim \mathrm{Unif}([0,1]\times[0,1]\times \ldots \times[0,1])$.

Example: A random point in a unit cube $[0,1]\times[0,1]\times[0,1]$ is obtained by three calls to a pseudo random number generator (Rnd_1, Rnd_2, Rnd_3).

Question: What is the chance that the point is inside a sphere of radius 1/3 centered at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$?



Solution: The chance is equal to the volume of the ball:

$4/3\ \pi\ (1/3)^3 = 4\pi/81.$

Multi-dimensional integration

Exercises:

$$A = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

$$\int_A (xy)^2 dxdy = ?$$

$$\int_A (x+y)^2 dxdy = ?$$

Multi-dimensional integration

Exercises:

$$A = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$$

$$\int_A (xy)^2 dxdy = ?$$

$$\int_A (x + y)^2 dxdy = ?$$

Multi-dimensional integration

Exercises:

$$A = \{-\infty < y < x < \infty\}$$

$$\int_A xye^{-(x^2+y^2)}dxdy = ?$$

Multi-dimensional integration

Exercises:

$$A = \{x^2 + y^2 \leq 1\}$$

$$\int_A (x+y)^2 dxdy = ?$$