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Follows Jim Pitman's  
book:  
Probability  
Sections 6.1-6.2

## # of Heads in a Random # of Tosses

- Suppose a fair die is rolled and let  $N$  be the number on top.



$$N=5$$

- Next a fair coin is tossed  $N$  times and  $H$ , the number of heads is recorded:



$$H=3$$

### Question:

Find the distribution of  $H$ , the number of heads?

# # of Heads in a Random Number of Tosses

## Solution:

- The conditional probability for the  $H=h$ , given  $N=n$  is

$$P(H = h|N = n) = \binom{n}{h} \left(\frac{1}{2}\right)^n, \quad \binom{n}{h} = 0 \text{ if } n < h.$$

- By the rule of the average conditional probability and using  $P(N=n) = 1/6$

$$P(H = h) = \sum_{n=1}^6 P(H = h|N = n)P(N = n) = \frac{1}{6} \sum_{n=1}^6 \binom{n}{h} \left(\frac{1}{2}\right)^n.$$

h	0	1	2	3	4	5	6
P(H=h)	63/384	120/384	99/384	64/384	29/384	8/384	1/384

# Conditional Distribution of $Y$ given $X = x$

Def: For each value  $x$  of  $X$ , the set of probabilities

$$P(Y=y|X=x)$$

where  $y$  varies over all possible values of  $Y$ , form a probability distribution, depending on  $x$ , called the conditional distribution of  $Y$  given  $X=x$ .

## Remarks:

- In the example  $P(H = h | N = n) \sim \text{binomial}(n, \frac{1}{2})$ .
- The unconditional distribution of  $Y$  is the average of the conditional distributions, weighted by  $P(X=x)$ .

# Conditional Distribution of $X$ given $Y=y$

**Remark:** Once we have the conditional distribution of  $Y$  given  $X=x$ , using **Bayes' rule** we may obtain the conditional distribution of  $X$  given  $Y=y$ .

• **Example:** We have computed distribution of  $H$  given  $N = n$ :

$$P(H = h|N = n) = \binom{n}{h} \left(\frac{1}{2}\right)^n.$$

• Using the product rule we can get the joint distr. of  $H$  and  $N$ :

$$P(H = h \& N = n) = P(H = h|N = n)P(N = n) = \frac{1}{6} \binom{n}{h} \left(\frac{1}{2}\right)^n,$$

• Finally:

$$\begin{aligned} P(N = n|H = h) &= \frac{P(H = h \& N = n)}{P(H = h)}; \\ &= \frac{\frac{1}{6} \binom{n}{h} \left(\frac{1}{2}\right)^n}{\frac{1}{6} \sum_{i=1}^6 \binom{i}{h} \left(\frac{1}{2}\right)^i} = \frac{\binom{n}{h}}{\sum_{i=1}^6 \binom{i}{h} 2^{n-i}}. \end{aligned}$$

# Number of 1's before the first 6.

Example: Roll a die until 6 appears.

Let  $Y = \#$  1's and  $X = \#$  of Rolls.

Question: What is the Distribution of  $Y$ ?

Solution:

1 - Find distribution of  $X$ :

$$P(X = x) = \left(\frac{5}{6}\right)^{x-1} \frac{1}{6}, \quad \text{so } X \sim \text{Geom}(1/6).$$

2- Find conditional distribution of  $Y$  given  $X=x$ .

$$P(Y = y | X = x) = \binom{x-1}{y} \left(\frac{1}{5}\right)^y \left(\frac{4}{5}\right)^{x-1-y},$$

so  $P(Y | X=x) \sim \text{Bin}(x-1, 1/5)$ .

## Number of 1's before the first 6.

3- Use the **rule of average conditional probabilities** to compute the unconditional distribution of  $Y$ .

$$\begin{aligned} P(Y = y) &= \sum_{x=y+1}^{\infty} P(Y = y|X = x)P(X = x) \\ &= \sum_{x=y+1}^{\infty} \binom{x-1}{y} \left(\frac{1}{5}\right)^y \left(\frac{4}{5}\right)^{x-1-y} \left(\frac{5}{6}\right)^{x-1} \frac{1}{6}; \\ &= \frac{1}{6} \left(\frac{1}{4}\right)^y \sum_{x=y+1}^{\infty} \binom{x-1}{y} \left(\frac{4}{6}\right)^{x-1}; \\ &= \frac{1}{6} \left(\frac{1}{4}\right)^y \frac{1}{y!} \left(\frac{4}{6}\right)^y \sum_{x=y+1}^{\infty} (x-1)(x-2)\dots(x-y) \left(\frac{4}{6}\right)^{x-y-1}; \\ &= \left(\frac{1}{6}\right)^{y+1} \frac{1}{y!} \sum_{x=y}^{\infty} (x)(x-1)\dots(x-y+1) \left(\frac{4}{6}\right)^{x-y}; \\ &= \left(\frac{1}{6}\right)^{y+1} \frac{1}{y!} \left[ \frac{d^y}{dr^y} \sum_{x=0}^{\infty} (r)^x \right]_{r=\frac{4}{6}} = \left(\frac{1}{6}\right)^{y+1} \frac{1}{y!} \left[ \frac{d^y}{dr^y} \frac{1}{1-r} \right]_{r=\frac{4}{6}}; \\ &= \left(\frac{1}{6}\right)^{y+1} \frac{1}{y!} y! \left(1 - \frac{4}{6}\right)^{-(y+1)} = \left(\frac{1}{2}\right)^{y+1} \end{aligned}$$

So  $Y = G-1$ , where  
 $G = \text{Geom}(1/2)$

# Conditional Expectation Given an Event

Definition: The conditional expectation of  $Y$  given an event  $A$ , denoted by  $E(Y|A)$ , is the expectation of  $Y$  under the conditional distribution given  $A$ :

$$E(Y|A) = \sum_{\text{all } y} y P(Y=y|A).$$

**Example:** Roll a die twice; Find  $E(\text{Sum} | \text{No 6's})$ .

$X+Y$	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$E(X+Y | \text{No 6's}) =$$

$$(2*1 + 3*2 + 4*3 + 5*4 + 6*5 + 7*4 + 8*3 + 9*2 + 10*1) / 25$$

$$= 6$$

# Linearity of Conditional Expectation

Claim:

For any set  $A$ :

$$E(X + Y | A) = E(X|A) + E(Y|A).$$

Proof:

$$\begin{aligned} E(X + Y | A) &= \sum_{\text{all } (x,y)} (x+y) P(X=x \ \& \ Y=y | A) \\ &= \sum_{\text{all } x} x \sum_{\text{all } y} P(X=x \ \& \ Y=y | A) \\ &\quad + \sum_{\text{all } y} y \sum_{\text{all } x} P(Y=y \ \& \ X=x | A) \\ &= \sum_{\text{all } x} x P(X=x | A) \\ &\quad + \sum_{\text{all } y} y P(Y=y | A) \\ &= E(X|A) + E(Y|A). \end{aligned}$$

# Using Linearity for 2 Rolls of Dice

**Example:** Roll a die twice; Find  $E(\text{Sum} \mid \text{No 6's})$ .

$$\begin{aligned} E(X+Y \mid X \neq 6 \ \& \ Y \neq 6) &= E(X \mid X \neq 6 \ \& \ Y \neq 6) + E(Y \mid X \neq 6 \ \& \ Y \neq 6) \\ &= 2 E(X \mid X \neq 6) \\ &= 2(1 + 2 + 3 + 4 + 5)/5 \\ &= 6 \end{aligned}$$

# Conditional Expectation

Claim: For any  $Y$  with  $E(Y) < \infty$  and any discrete  $X$ ,

$$E(Y) = \sum_{\text{all } x} E(Y|X=x) P(X=x).$$

This is called the "rule of average conditional expectation".

Definition of conditional expectation:

The conditional expectation of  $Y$  given  $X$ , denoted by  $E(Y|X)$ , is a random variable that depends on  $X$ . Its value, when  $X = x$ , is  $E(Y|X=x)$ .

So if  $Z = E[Y | X]$  then

$$P[Z = z] = \sum \{ P(x) : E(Y | X = x) = z \}$$

Compact form of rule of average conditional E's :

$$E[Y] = E[E(Y|X)]$$

# Conditional Expectations: Examples

## Example:

Let  $N$  = number on a die,  $H$  = # of heads in  $N$  tosses.

**Question:** Find  $E(H|N)$ .

**Solution:**

We have seen that  $P(H|N=n) \sim \text{Bin}(n, \frac{1}{2})$ . So  $E(H|N=n) = n/2$ .  
and therefore  $E(H|N) = N/2$ .

**Question:** Find  $E(H)$ .

**Solution:** This we can do by conditioning on  $N$ :

$$E(H) = E(E(H|N)) = E(N/2) = 3.5/2 = 1.75.$$

# Conditional Expectation - Examples

Roll a die until 6 comes up.

$N$  = # of rolls;  $Y$  = # of 1's.

**Question:** Compute  $E[Y]$ .

**Solution:** We will condition on  $N$ :  $E[Y] = E[E[Y|N]]$ .

**Then:**  $E[Y|N=n] = (N-1)/5$ .

$N \sim \text{Geom}(1/6)$  and  $E[N] = 6$ . Thus

$$E[Y] = E[E[Y|N]] = E[(N-1)/5] = 1.$$

**Note:** We've seen before then  $Y = G - 1$  where

$G \sim \text{Geom}(1/2)$ . This also gives:  $E[Y] = 2-1 = 1$ .

# Conditional Expectation

Roll a die until 6 comes up.

$S$  = sum of the rolls;  $N$  = # of rolls.

**Question:** Compute  $E[S]$ ,

**Solution:** We will condition on  $N$ :  $E[S] = E[E[S|N]]$ .

**Need:**  $E[S|N=n]$ ;

Let  $X_i$  denote the # on the  $i^{\text{th}}$  roll die.

$E[S|N=n] = E(X_1 + \dots + X_n) = 3.5n$ ; so  $E[S|N] = 3.5N$ .

$N \sim \text{Geom}(1/6)$  and  $E[N] = 6$ . Thus

$$E[S] = 3.5E[N] = 3.5 * 6 = 21.$$

# Success counts in overlapping series of trials

## Example:

Let  $S_n$  number of successes in  $n$  independent trials with probability of success  $p$ .

**Question:** Calculate  $E(S_m | S_n = k)$  for  $m < n$

## Solution:

$S_m = X_1 + \dots + X_m$ , where  $X_j$  is indicator of success on  $j^{\text{th}}$  trial.

$$E(S_m | S_n = k) = \sum_{j=1}^m E(X_j | S_n = k)$$

$$E(X_j | S_n = k) = P(j^{\text{th}} \text{ trial is a success} | S_n = k)$$

$$= \frac{P(j^{\text{th}} \text{ trial is a success, } k-1 \text{ of other } n-1 \text{ trials are successes)}}{P(S_n = k)}$$

$$= \frac{p \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{k}{n}; \quad E(S_n | S_m = k) = \frac{mk}{n}.$$

# Conditional Expectation

**Comment:** Conditioned variables can be treated as constants when taking expectations:

$$E[g(X,Y)|X=x] = E[g(x,Y)|X=x]$$

**Example:**

$$E[XY|X=x] = E[xY|X=x] = xE[Y|X=x];$$

so as random variables:

$$E[XY|X] = XE[Y|X].$$

**Example:**  $E[aX + bY|X] = aX + bE[Y|X];$

**Question:** Suppose  $X$  and  $Y$  are independent, find  $E[X+Y|X]$ .

**Solution:**  $E(X+Y|X) = X + E(Y|X) = X + E(Y)$ , by independence.