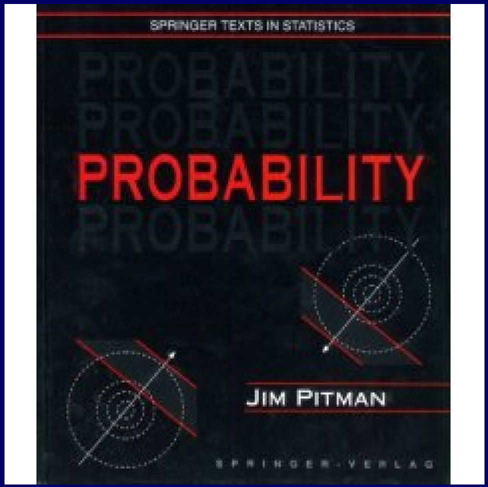
Introduction to probability

Stat 134 FAll 2005

Berkeley



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Follows Jim Pitman's book:

Probability

Section 4.5

Cumulative Distribution Function

- Definition: For a random variable X, the function

$$F(x) = P(X \cdot x),$$

is called the cumulative distribution function (cdf).

- A distribution is called continuous when the cdf is continuous.

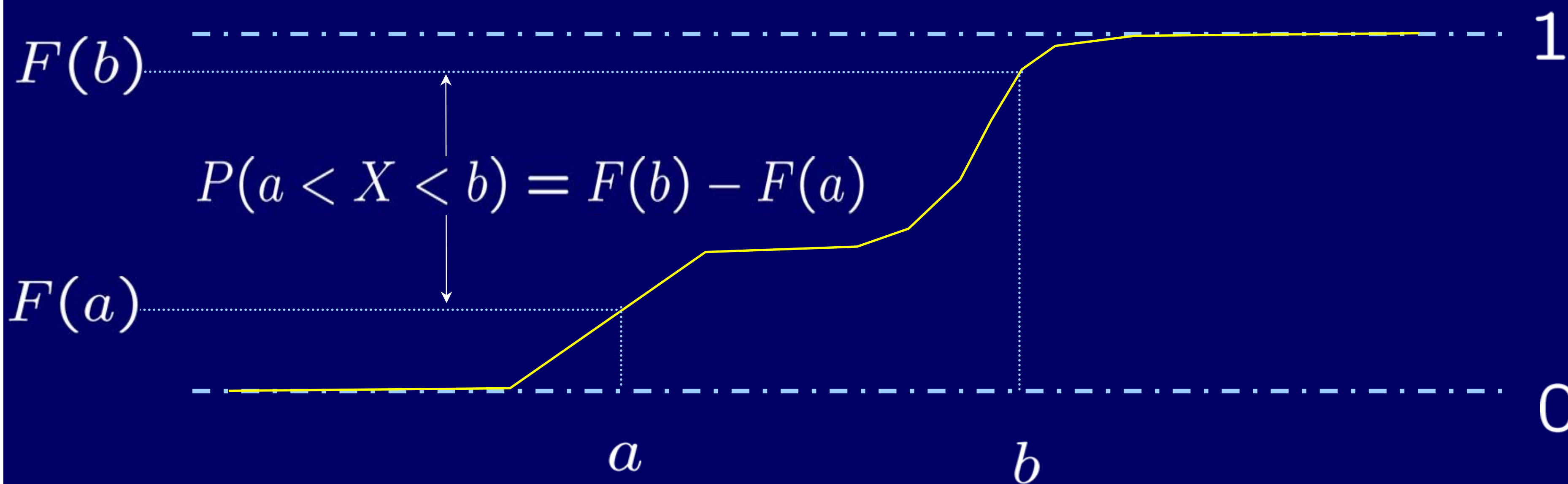
Properties of the CDF.

Claim: The cdf determines the probability of lying inside intervals: P(a < X· b) = F(b) – F(a).

- P{(a , b]} = F(b) – F(a).
- P{(a , b)} = $\lim_{x \to b^-}$ F(x) – F(a).
- P{(a,b) Å (c,d) } = P{(c,b)}, for a<c<b<d.
- P{(a,b) ∪ (c,d) } = P{(a,d)}, for a<c<b<d.
- P{b} = P{(a,b]} – P{(a,b)} = F(b) - $\lim_{x \to b^-}$ F(x).

If F is continuous, P(a,b) = F(b) – F(a) and F(x) = 0 for all x.

Cumulative Distribution Function

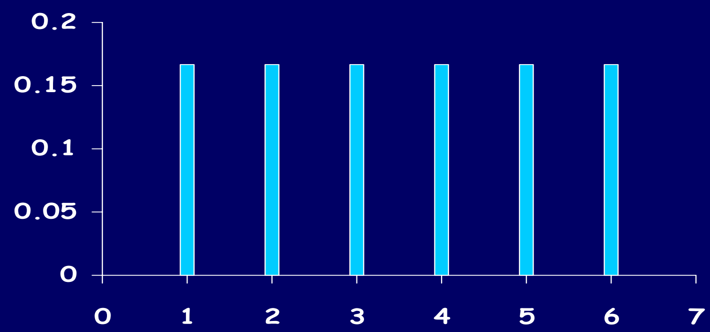


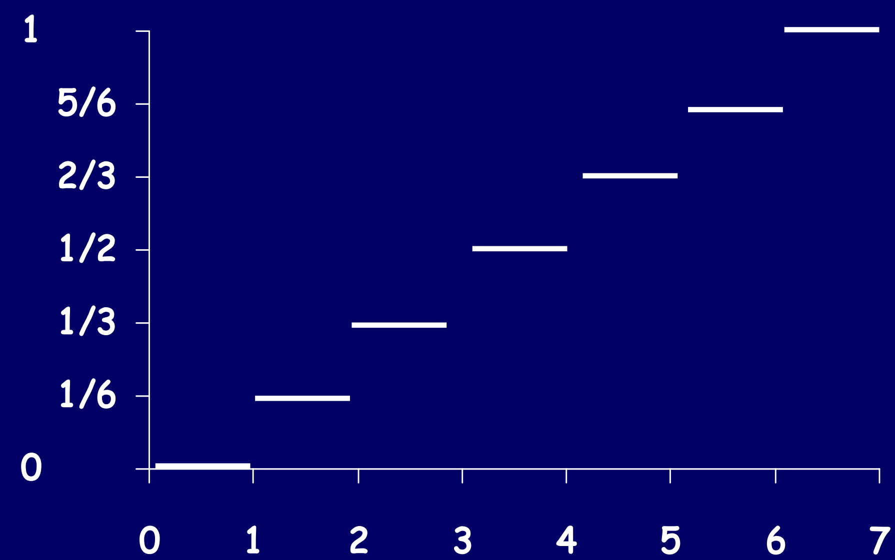
Discrete Distributions

Claim: If X is a discrete variable, then

$F(x) = \sum_{y \leq x} P(X=y)$.

- Example: Let X be the numbers on a fair die.





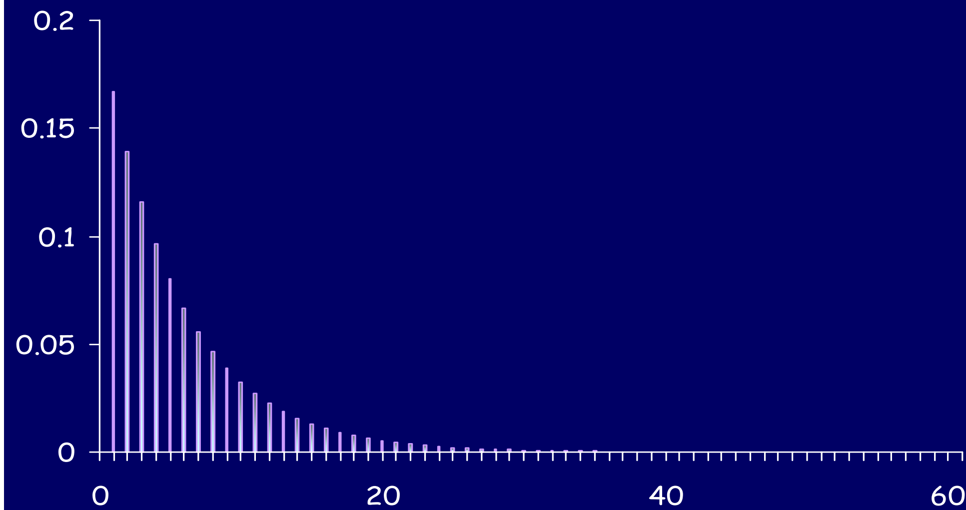
$P\{x\} = F(x) - F(x^-)$: $P\{X=6\} = 1 - 5/6 = 1/6$.

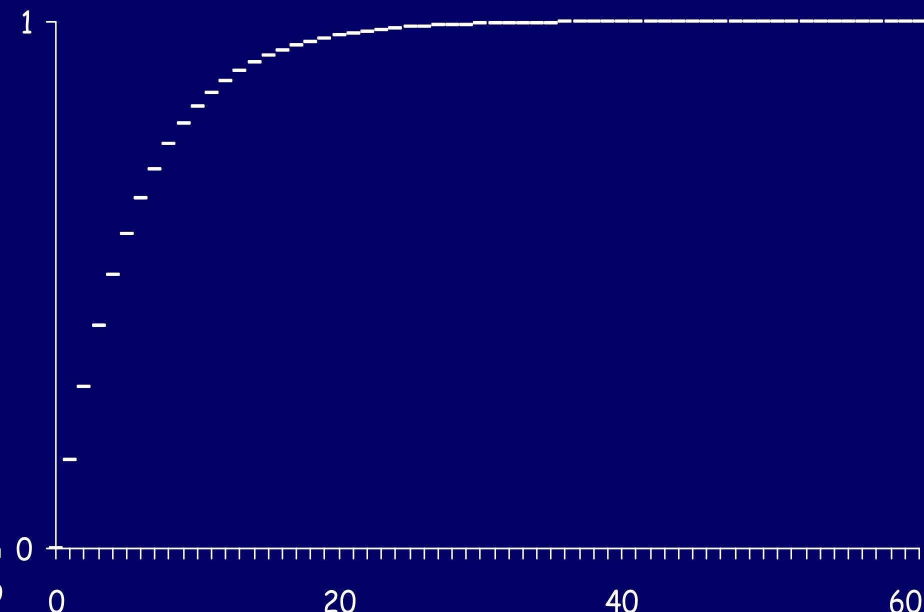
Discrete Distributions

• Example: Let X be the number of rolls before the first six appears when rolling a fair die.

$X \sim$ Geom(1/6), $P(n)=(5/6)^{n-1}\,1/6$ (for integer n)

$F(x) = 1 - (5/6)^{\lfloor x \rfloor}$





For integer x:

$P\{n\} = F(n) - F(n\text{-}) = 1 - (5/6)^n - (1 - (5/6)^{n-1}) = (5/6)^{n-1}\,1/6.$

Continuous Distributions with a Density

- Claim: If X is a continuous variable with a density $f_X(x)$ then $F(x) = \int_{-\infty}^{x} f_X(y)dy$.

- Claim: if X has a continuous distribution given by a c.d.f $F(x)$ and
- F is differentiable in all but finitely many points,
- Then the distribution has a density $f_X(x) = F'(x)$.

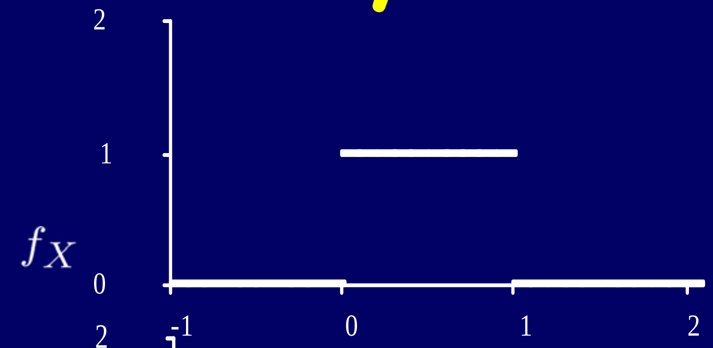
"Pf": Similarly to the discrete case

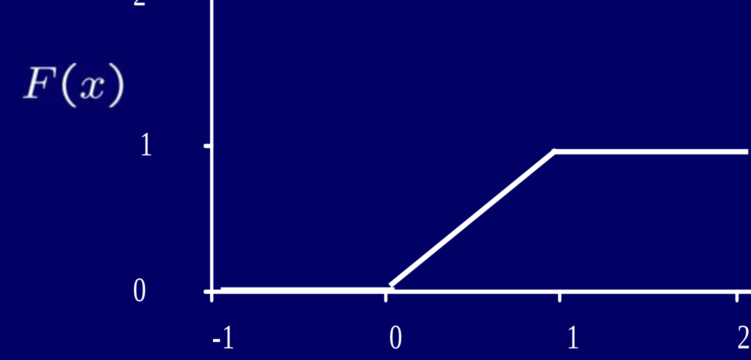
- $F'(x)dx = F(x+dx) - F(x) = P(x < X < x + dx) = f(x)dx$
- so $F'(x) = f(x)$.

Distributions with a Density

- Example: U ~ Uniform(0,1).

$$f_X(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases} \qquad F(x) = \begin{cases} x, & \text{if } 0 \le x \le 1 \\ 0, & \text{if } x < 0 \\ 1, & \text{if } x > 1. \end{cases}$$





Question: Show that X = 2|U – ½| is a Uniform variable as well.

Solution: P(X· x) = P(2|U – ½|· x) = P(½ - x/2 · U · ½ + x/2)
=F(1/2 + x/2) – F(1/2 –x/2) = 1/2 + x/2- 1/2 + x/2
=x = F(x).

So X has the same cdf as U therefore the same distribution.

Question: What is the distribution of 4| |U-1/2| - 1/4| ?

Distribution of Min and Max.

Let $X_1, X_2, \ldots, X_n$ be independent random variables.

$X_{max} = \max\{X_1, X_2, \ldots, X_n\}$ & $X_{min} = \min\{X_1, X_2, \ldots, X_n\}$.

It's easy to find the distribution of X_{max} and X_{min} by using the cdf's by using the following observations:

For any x

- $X_{max} \cdot x$ iff $X_i \cdot x$ for all i.
- $X_{min} > x$ iff $X_i > x$ for all i.

Distribution of Min and Max.

$F_{max}(x) = P(X_{max} \le x) = P(X_1 \le x, X_2 \le x, \dots X_n \le x)$

$\quad = P(X_1 \le x) P(X_2 \le x) \dots P(X_n \le x)$, by indep.

$\quad = F_1(x) F_2(x) \dots F_n(x).$

$F_{min}(x) = P(X_{min} \le x) = 1 - P(X_{min} > x)$

$\quad = 1 - P(X_1 > x, X_2 > x, \dots X_n > x)$

$\quad = 1 - (1 - F_1(x))(1 - F_2(x)) \dots (1 - F_n(x)).$

- $F_{max}(x) = F_1(x) F_2(x) \dots F_n(x).$
- $F_{min}(x) = 1 - (1 - F_1(x))(1 - F_2(x)) \dots (1 - F_n(x)).$
- Example: Suppose X_i's all have exponential distributions with rates λ_i.

Question: Find the distribution of X_{min}.

Solution: For $x < 0$, $F_i(x) = 0$ and for $x \geq 0$, $F_i = 1 - e^{-\lambda x}$.

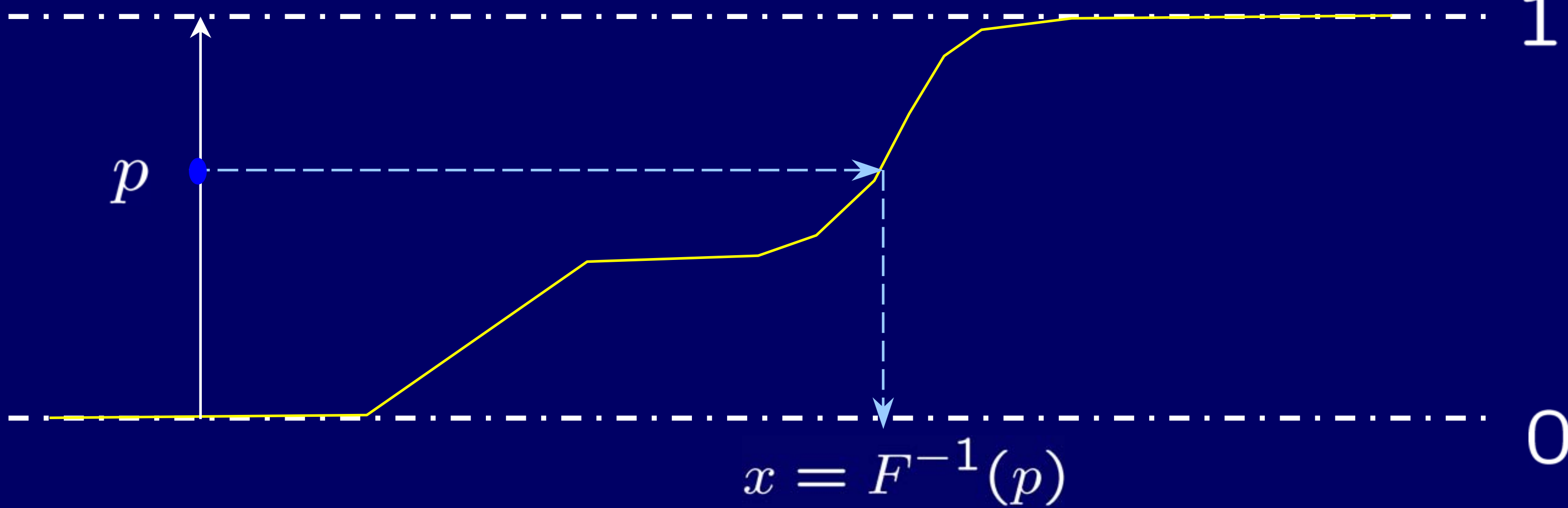
So for $x < 0$, $F_{min} = 0$ and for $x \geq 0$, $F_{min} = 1 - e^{-\lambda_1 + \lambda_2 + \dots + \lambda_n}$.

$X_{min} \sim Exp(\lambda_1 + \lambda_2 + \dots \lambda_n\}$.

Percentiles and Inverse CDF.

Definition: The p'th quantile of the distribution of a random variable X is given by the number x such that

$$P(X \cdot x) = p.$$



Simulating a RV with a given distribution

Example:

A pseudo random number generator generates a sequence $U_1, U_2, \ldots U_n$ each between 0 and 1, which are i.i.d. Uniform(0,1) random variables.

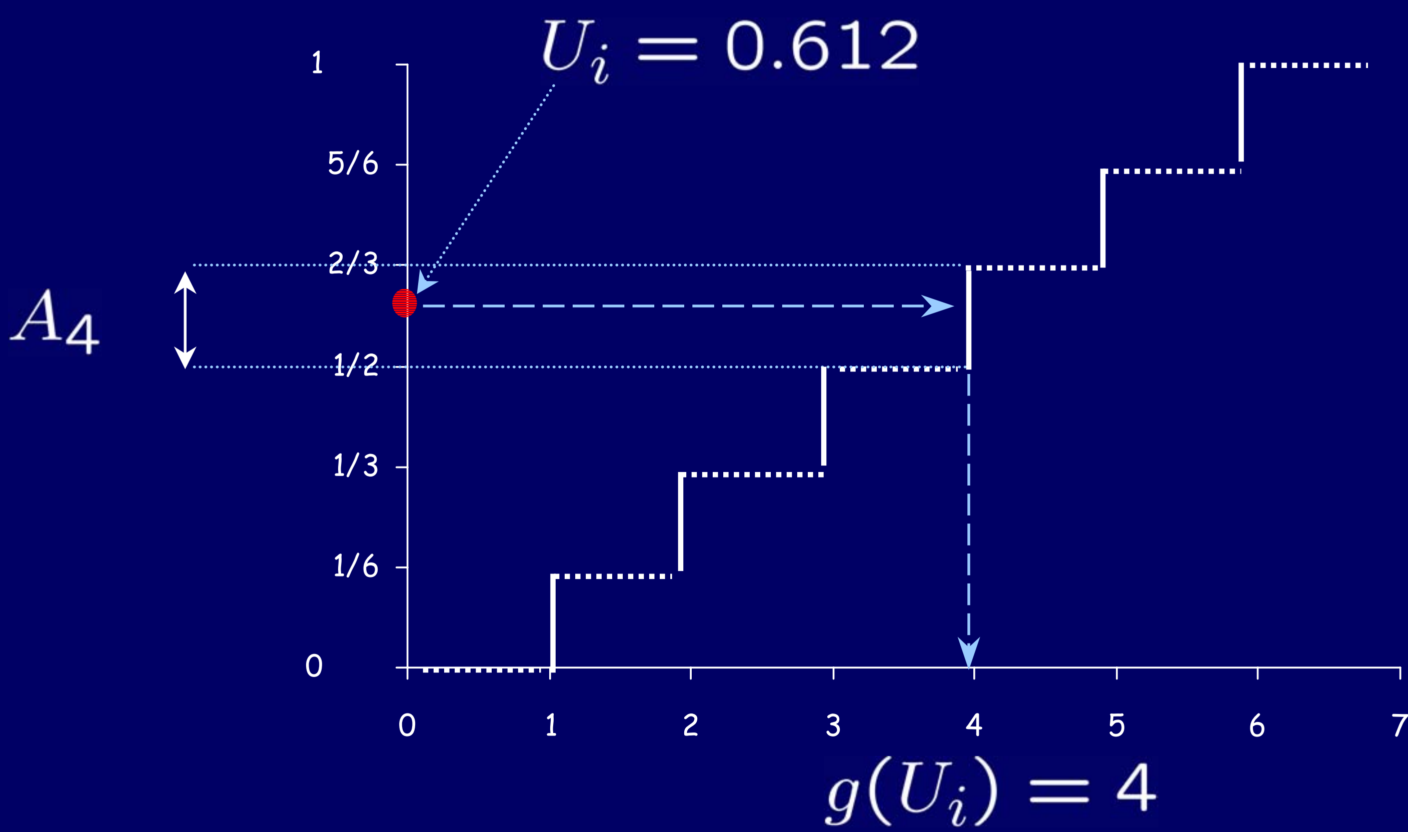
Question: You are working for a company that is developing an on-line casino. How can you use this sequence to simulate n rolls of a fair die?

Solution: You need a new sequence of variables each with a discrete uniform distribution on 1,2,3,4,5,6.

Simple idea -- break up the unit interval into 6 intervals $A_k = ((k-1)/6, k/6]$, each of length 1/6 & let

$$g(u) = k \text{ if } u \in A_k.$$

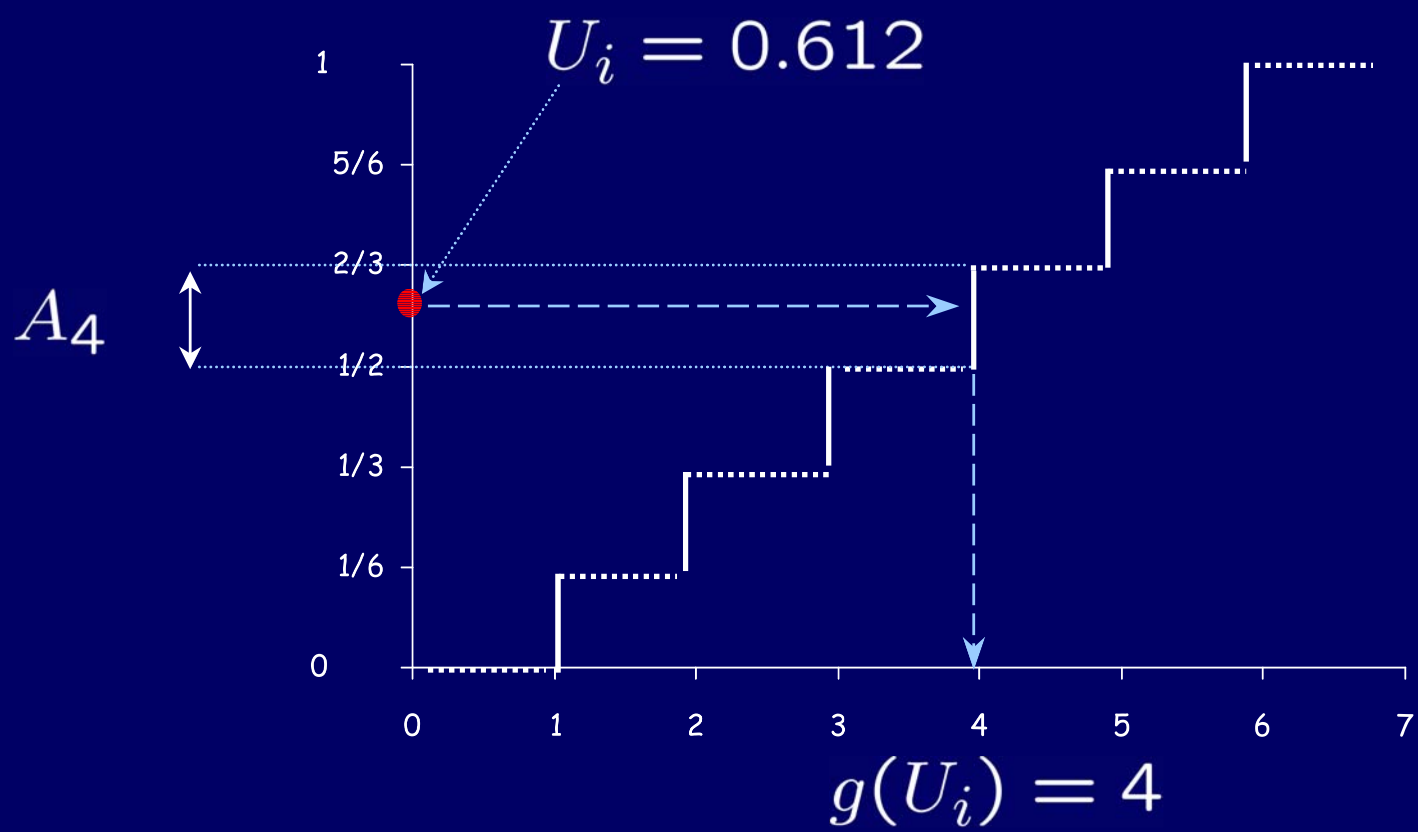
Simulating a RV with a given distribution



Then $X_i = g(U_i)$ each has the desired distribution:

$$P(X_i = k) = P(g(U_i)=k) = P(A_k) = 1/6.$$

Simulating a RV with a given distribution



Note that for i=1,2,3,4,5,6 it holds that:

g(F(i)) = g(i/6)=i.

Simulating a RV with a given distribution

For a general discrete random variable with $P(k) = p_k$, $k=1,2,3\ldots$ we can use the same idea, except each interval A_k is now of length p_k :

$$A_k = (p_1+p_2+\ldots+p_{k-1}\ ,\ \ p_1+p_2+\ldots+p_{k-1}+p_k].$$

The function $g(u)$ defined in this way is a kind of an inverse of the cdf:

$$g(F(k)) = k \text{ for all } k=1,2,3,$$

This observation can be used to simulate continuous random variables as well.

Define $F^{-1}(x) = \min_y F(y) = x$.

Inverse cdf Applied to Standard Uniform

Define $F^{-1}(x) = \min\{ y : F(y) = x\}$

Theorem: For any cumulative distribution function F, with inverse function F^{-1}, if U has a Uniform (0,1) distribution, then a random variable $G = F^{-1}(U)$ has F as a cdf.

Proof (imagine first that F is strictly increasing):

$$P(G \cdot x) = P\{U \cdot F(x)) = F(x).$$

Simulating a RV with a given distribution

A pseudo random number generator gives a sequence $U_1, U_2, \ldots U_n$ each between 0 and 1, which behave like a sequence of Uniform(0,1) random variables.

Question: You are working for an engineering company that wants to model failure of circuits. You may assume that a circuit has n components whose life-times have $Exp(\lambda_i)$ distribution. How can you generate n variables that would have the desired distributions?

Solution: The cdf's are $F_i(x) = 1 - e^{-\lambda_i x}$.

So we let $G_i = G(U_i)$ where $G(u) = -\log(1-u)/\lambda_i$.