The Threshold Value for the Planted Partition Model

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Based on a joint work with:

1. Joe Neeman (U.C. Berkeley)
2. Allan Sly (U.C. Berkeley)

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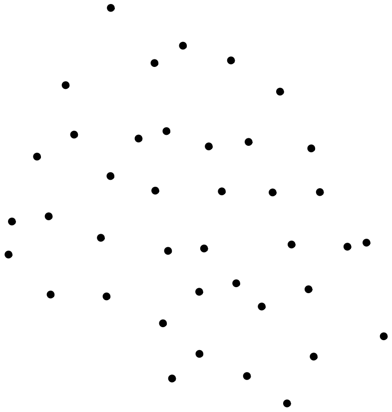

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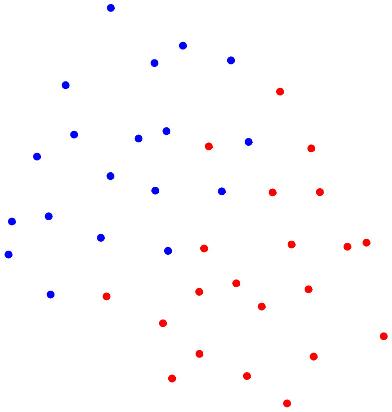

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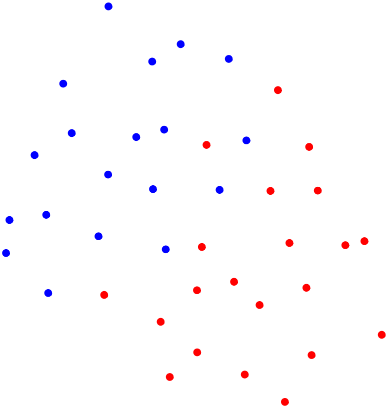

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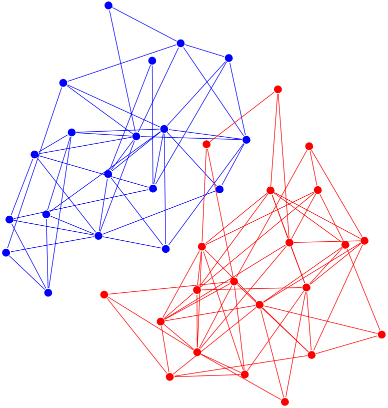

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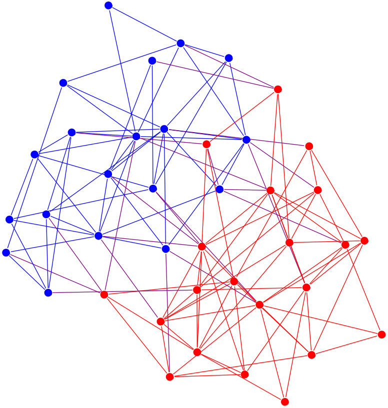

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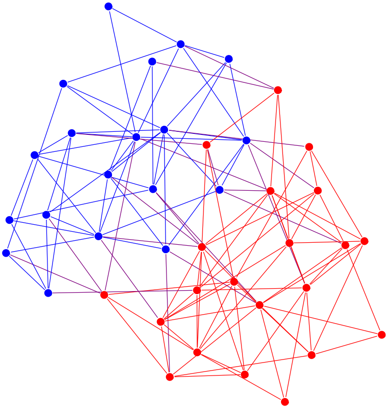

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Variations: more than two classes, un-balanced classes, d-regular, ...

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Clustering network data. This model was introduced in statistics by Holland-Laskey-Leinhardt '83. Both parameter estimation and label recovery are studied.

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Snijders-Nowicki '97	ML, EM, etc.	$a - b = \Omega(n)$
Bickel-Chen '09	G-N modularity	$\frac{a-b}{\sqrt{a+b}} = \Omega(\log n)$
Chatterjee-Rohe-Yu '10	spectral	$a - b = \Omega(n)$
Choi-Wolfe-Airoldi '10	ML	$a + b = \Omega(\log^3 n)$, $\frac{a-b}{\sqrt{a+b}} = \Omega(1)$

Many, many more algorithms without performance guarantees (survey in Lancichinetti-Fortunato '09).

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- Well done!

The sparse case: a phase transition

Decelle-Krzakala-Moore-Zdeborová '11: "recovery" means getting a partition that is positively correlated with the truth.

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If

$$\frac{(a-b)^2}{2(a+b)} > 1$$

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"Physics" proof using belief propagation.

Non-Reconstruction results of M-Neeman-Sly

Theorem

If $a+b>2$ and $(a-b)^2 \leq 2(a+b)$ then, for any fixed vertices u and v,

$$\mathbb{P}_n(\sigma_u = + | G, \sigma_v = +) \to \frac{1}{2}$$

$\implies$ *impossible to recover a partition that is correlated with the true partition.*

Theorem

Let $\mathbb{P}'_n$ be the law of $G(n, \frac{a+b}{2n})$. If $(a-b)^2 < 2(a+b)$ then $\mathbb{P}_n$ and $\mathbb{P}'_n$ are mutually contiguous i.e., for a sequence of events A_n, $\mathbb{P}_n(A_n) \to 0$ if, and only if, $\mathbb{P}'_n(A_n) \to 0$.

Reconstruction Results of M-Neeman-Sly

Theorem

Assume $(a-b)^2 > 2(a+b)$.

- *The parameters* a, b *are identifiable.*
- *Let* $\mathbb{P}'_n$ *be the law of* $G(n, \frac{a+b}{2n})$. *Then* $\mathbb{P}_n$ *and* $\mathbb{P}'_n$ *are asymptotically orthogonal. In other words, there exist events* A_n *such that* $\mathbb{P}_n(A_n) \to 1$ *and* $\mathbb{P}'_n(A_n) \to 0$.

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- Still approximately correct for small enough neighborhoods.

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$$X_{k,n} \to Pois\left(\frac{1}{k2^{k+1}}((a+b)^k + (a-b)^k)\right).$$

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can detect the difference in mean.

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then $X_{n,k}$ and $Y_{n,k}$ are almost orthogonal.

Contiguous with $G(n, \frac{a+b}{2}n)$ when $(a-b)^2 < 2(a+b)$

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- Most of the work is devoted to apply the "small graph conditioning method" to show that $\lim_{\epsilon\to 0}\lim_{n\to\infty}\mathbb{P}(Y_n < \epsilon) = 0$.

Morals and open problems

- Cluster information is detectible even in cases where it is impossible to identify with certainty the cluster identity of any individual node.
- Algorithms such as Belief Propagation are likely to be effective in detecting community structures.
- Challenge: efficient algorithm for community detection.
- Challenge: Extension to other models.