Robust Dimension Free Isoperimetry in Gaussian Space

Elchanan Mossel and Joe Neeman (UC Berkeley)

May 10, 2012
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The Gaussian Isoperimetric problem

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- The inequality states that if $A \subset \mathbb{R}^n$ and $B = \{x \in \mathbb{R}^n : x \cdot a \geq b\} \subset \mathbb{R}^n$ is a half-space of the same gaussian measure ($\gamma_n(A) = \gamma_n(B)$) then:

\[
\gamma_n^+\left(\mathbb{R}^n \setminus A\right) \geq I\left(\gamma_n\left(\mathbb{R}^n \setminus B\right)\right),
\]

where $I(x) := \phi(\Phi^{-1}(x))$ and $\phi, \Phi$ are the Gaussian density, CDF.)
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  \[ \gamma_n^+(A) := \liminf_{\epsilon \to 0} \frac{1}{\epsilon} (\gamma_n(A_\epsilon) - \gamma_n(A)), \quad A_\epsilon = \{ y : d_2(y, A) \leq \epsilon \}. \]
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- In other words: $\gamma_n^+(A) \geq I(\gamma_n(A))$, where $I(x) := \varphi(\Phi^{-1}(x))$ and $\varphi, \Phi$ are the Gaussian density, CDF).
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Are half spaces the **only** minimizers of Gaussian surface area?
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- **Cianchi, Fusco, Maggi, and Pratelli (2011):** If $\gamma_n^+(A) \leq I(\gamma_n(A)) + \delta$ then there exists a half space $B$ with $\gamma_n(A \Delta B) \leq c(n)\delta^{1/2}$.

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- **M, Neeman (12):** If $\gamma_n^+(A) \leq I(A) + \delta$ then there exists a half space with $\gamma_n(A \Delta B) \leq C \log^{-1/6}(1/\delta)$. 

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  \[\gamma_n(A \Delta B) \leq C \log^{-1/6}(1/\delta)\]

- **Natural conjecture:** Exists a half space $B$ with
  \[\gamma_n(A \Delta B) \leq C \sqrt{\delta}\]
A common approach is using geometric techniques such as symmetrizations (starting with Steiner 1838!).
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Our approach follows Bobkov and Ledoux in:
- Analyzing a function version of the inequality.
- Utilizing the semi-group flow.
Bobkov’s inequality

Bobkov proved a functional version of the inequality:

- **Bobkov**: For any smooth function $f : \mathbb{R}^n \rightarrow [0, 1]$ of bounded variation,

$$I(\mathbb{E}f) \leq \mathbb{E}\sqrt{I^2(f)} + \|\nabla f\|_2^2.$$
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  \[
  I(\mathbb{E} f) \leq \mathbb{E} \sqrt{I^2(f) + \|\nabla f\|_2^2}.
  \]

- Since \( I(0) = I(1) = 0 \), then one can show that if \( A \) is a "nice set" then:
  \[
  I(\gamma_n(A)) \leq "\mathbb{E}[\|\nabla 1_A\|_2]" = \gamma_n^+(A)
  \]
Consider the Ornstein-Uhlenbeck semigroup:

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}} y) \, d\gamma_n(y).$$
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look at: \(\psi(t) := \mathbb{E}\sqrt{I^2(P_t f) + \|\nabla P_t f\|^2}.\)
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Suffices to prove $\psi_t$ is decreasing.
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Suffices to prove \(\psi_t\) is decreasing.

Nice properties that allow to establish \(\psi'(t) \leq 0\):

- \(II'' = -1\)
- Integration by parts \(\int -fLg \, d\gamma_n = \int \langle \nabla f, \nabla g \rangle d\gamma_n\) (where \(Lf(x) = \Delta f(x) - \langle x, \nabla f \rangle\) is the generator).
- etc.
Carlen and Kerce analysis

Carlen and Kerce (2001):

Let $f : \mathbb{R}^n \to [0, 1]$ be smooth. Define $h_t = \Phi^{-1} \circ (P_t f)$ and

$$\delta(f) = E \sqrt{I_2(f) + \|\nabla f\|^2} - I(E f).$$

Then

$$\delta(f) \geq \int_0^\infty E \phi(h_t) \|H(h_t)\|_F^2 (1 + \|\nabla h_t\|^2)^{3/2} dt,$$

where $H(h_t)$ is the Hessian matrix of $h_t$ and $\|\cdot\|_F$ denotes the Frobenius norm.

If $\delta(f) = 0$, then $h_t$ is linear for $t > 0$, implying that $P_t f$ is Gaussian for all $t$.

If $f = 1$ and $\delta(f) = 0$, then $f$ is a half-space.
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Let $f : \mathbb{R}^n \rightarrow [0, 1]$ be smooth. Define $h_t = \Phi^{-1} \circ (P_t f)$ and

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Let \( f : \mathbb{R}^n \to [0, 1] \) be smooth. Define \( h_t = \Phi^{-1} \circ (P_t f) \) and

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Then

\[
\delta(f) \geq \int_0^\infty \mathbb{E} \frac{\varphi(h_t)\|H(h_t)\|_F^2}{(1 + \|\nabla h_t\|^2)^{3/2}} \, dt,
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where \( H(h_t) \) is the Hessian matrix of \( h_t \) and \( \| \cdot \|_F \) denotes the Frobenius norm.
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\( \delta(f) = 0 \implies \) \( h_t \) is linear \( t > 0 \implies P_t f \) is Gaussian \( \forall t \).
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Then

$$\delta(f) \geq \int_{0}^{\infty} \mathbb{E} \frac{\varphi(h_t) \|H(h_t)\|_F^2}{(1 + \|\nabla h_t\|^2)^{3/2}} dt,$$

where $H(h_t)$ is the Hessian matrix of $h_t$ and $\| \cdot \|_F$ denotes the Frobenius norm.

- $\delta(f) = 0 \implies h_t$ is linear $t > 0 \implies P_t f$ is Gaussian $\forall t$.
- $f = 1_A$ and $\delta(f) = 0$ by limiting arguments $f$ is a half-space.
M + Neeman proof Strategy

- Use the Carlen and Kerce bound.
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Show that if \( \delta(f) \) is small then \( h_t \) close to linear.
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Show that if $\delta(f)$ is small then $h_t$ close to linear.
Conclude that $f$ is close to a Gaussian / half-space.
To demonstrate the main ideas of the proof assume a stronger result than Carlen and Kerce:

\[ \delta(f) \geq \int_0^\infty \mathbb{E}\|H(h_t)\|_F^2 \, dt, \]

\[ (h_t = \Phi^{-1} \circ (P_t f), \quad \delta(f) = \mathbb{E}\sqrt{l^2(f) + \|\nabla f\|^2} - l(\mathbb{E}f).) \]
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If \( \delta(f) < \epsilon^2 \) then there exists a \( t \in [0, \epsilon] \) with \( \|H(h_t)\|_F^2 \leq \epsilon \).
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If \(\delta(f) < \epsilon^2\) then there exists a \(t \in [0, \epsilon]\) with \(\|H(h_t)\|_F^2 \leq \epsilon\).

Second order Poincare inequality: For any twice-differentiable \(f \in L_2(\mathbb{R}^n, \gamma_n)\),

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\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 \leq \mathbb{E}\|H(f)\|_F^2,
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Now apply \( P_t^{-1} \) to obtain that \( f \) is close to Gaussian.
Challenges

For the real proof there are a few challenges:

- Need to prove a second order Poincare inequality.

- $P_{t-1}$ is not bounded - so cannot simply apply it.

- Main challenge: prove that there exists a $t^* \in (t, t-1)$ such that for $t > t^*$ and $f$ with $E_f \leq 1/2$ it holds that:

\[
E_{\phi}(h_t) \|H(h_t)\|_F^2 (1 + \|\nabla h_t\|_2^2)^{3/2} \geq c I_2 \left( E_f \right) (E(f(H(h_t))))^{4} \log^{-3/4} E_f.
\]

- Given (*) , if $\delta(f) < \epsilon$ then $
\int_{t^*}^{t-1} E_{\phi}(h_t) \|H(h_t)\|_F^2 (1 + \|\nabla h_t\|_2^2)^{3/2} dt < \epsilon.
\]

Therefore there exists $t < t^*$ such that $E(f(H(h_t))) \leq c^{-1/4} \epsilon^{1/4} \log^{-3/4} E_f$.

- So the main challenge is to prove (*).
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Given (*), if $\delta(f) < \epsilon$ then
$$\int_t^{t^* - 1} E \phi(h_t) \|H(h_t)\|_F^2 (1 + \|\nabla h_t\|_2^2)^{3/2} \geq c I^2 (E f) (E (\|H(h_t)\|_F^2))^{4 \log 3/4 1} E f I^{1/2}.$$ 

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\[
(*) \quad \mathbb{E} \frac{\varphi(h_t) \|H(h_t)\|_F^2}{(1 + \|\nabla h_t\|^2)^{3/2}} \geq cl^2(\mathbb{E}f) \left(\mathbb{E}(\|H(h_t)\|_F^2)\right)^4 \log^{-3} \frac{1}{\mathbb{E}f}.
\]

- Therefore there exists $t < t_\star$ such that $\mathbb{E}(\|H(h_t)\|_F^2) \leq c^{-1} \frac{1}{\mathbb{E}f}$.

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\]

- Given (*) , if $\delta(f) < \epsilon$ then

\[
\int_{t* - 1}^{t} \mathbb{E} \varphi(h_t)\|H(h_t)\|_F^2 \frac{dt}{(1 + \|\nabla h_t\|^2)^{3/2}} < \epsilon.
\]

Therefore there exists $t < t_*$ such that

\[
\mathbb{E}\left(\|H(h_t)\|_F^2\right) \leq c^{-1} \epsilon^{1/4} \log^{3/4} \frac{1}{\mathbb{E}f} l^{-2}(\mathbb{E}f).
\]
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- Main challenge: prove that there exists a $t_*$ such that for $t > t_* - 1$ and $f$ with $\mathbb{E} f \leq 1/2$ it holds that:

\[
(\star) \quad \mathbb{E} \frac{\varphi(h_t) \| H(h_t) \|_F^2}{(1 + \| \nabla h_t \|_2^2)^{3/2}} \geq c l^2(\mathbb{E} f) \left( \mathbb{E} (\| H(h_t) \|_F^2) \right)^4 \log^{-3} \frac{1}{\mathbb{E} f}.
\]

- Given (\star), if $\delta(f) < \epsilon$ then

\[
\int_{t_* - 1}^{t} \mathbb{E} \frac{\varphi(h_t) \| H(h_t) \|_F^2}{(1 + \| \nabla h_t \|_2^2)^{3/2}} dt < \epsilon.
\]

Therefore there exists $t < t_*$ such that

\[
\mathbb{E} (\| H(h_t) \|_F^2) \leq c^{-1} \epsilon^{1/4} \log^{3/4} \frac{1}{\mathbb{E} f} l^{-2}(\mathbb{E} f)
\]

- So the main challenge is to prove (\star).
A second order Poincare inequality

- **2nd order Poincare inequality**: For any twice-differentiable $f \in L_2(\mathbb{R}^n, \gamma_n)$,

$$ \min_{a,b} \mathbb{E}((f(x) - a \cdot x - b)^2) \leq \mathbb{E}\|H(f)\|_F^2, $$
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- \( \min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 = \sum_{|\alpha|\geq2} b_\alpha^2 \alpha! \).
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  $$\min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2 = \sum_{|\alpha| \geq 2} b_\alpha^2 \alpha!.$$

  \[
  \mathbb{E}\|H(f)\|_F^2 = \sum_{i,j} \mathbb{E}\left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 \\
  = \sum_{i \neq j} \sum_{\{\alpha: \alpha_i, \alpha_j \geq 1\}} b_\alpha^2 \alpha_i \alpha_j \alpha! + \sum_i \sum_{\{\alpha: \alpha_i \geq 2\}} b_\alpha^2 \alpha_i (\alpha_i - 1) \alpha! \\
  \geq \sum_{|\alpha| \geq 2} b_\alpha^2 \alpha! = \min_{a,b} \mathbb{E}(f(x) - a \cdot x - b)^2.
  \]
Boundedness of $P_t^{-1}$

- **Challenge**: Assume $\mathbb{E}(f_t - \Phi(a\dot{x} + b))^2 \leq \epsilon$. Is it true that

$$\min_{a \in [0, \infty]} \mathbb{E}(\|f_t - \Phi(a'\dot{x} + b)\|_2^2 \leq \epsilon'(t, \epsilon)?$$
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- **E.g. by Ledoux (94):** $\mathbb{E}f(f - P_tf) \leq c\sqrt{t\mathbb{E}\|\nabla f\|}$.
- **If $\mathbb{E}\|\nabla f\| \geq 10$ then**
  \[
  \delta(f) = \mathbb{E}\sqrt{l^2(f) + \|\nabla f\|^2} - l(\mathbb{E}f) \geq \mathbb{E}\|\nabla f\| - l(\mathbb{E}f) \geq 9.
  \]
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- **Similar arguments for sets.**
The main challenge

- Need to prove that for \( f \) taking values in \([0, 1]\) and \( \mathbb{E} f \leq 1/2 \):

\[
(*) \quad \mathbb{E} \frac{\varphi(h_t)\|H(h_t)\|^2_F}{(1 + \|\nabla h_t\|^2)^{3/2}} \geq c(\mathbb{E} f) \left( \mathbb{E}(\|H(h_t)\|^2_F) \right)^4
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\]

- For this, using the reverse log Sobolev inequality we prove that for \( t \) large enough:

\[
(**) \quad \|\nabla h_t\| \leq 1 \text{ a.s.} , \quad \|\nabla f_t\| \leq \frac{\sqrt{2e^{-t}}}{\sqrt{1 - e^{-2t}} f_t} \sqrt{\log \frac{1}{f_t}} \text{ a.s}
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The main challenge

- Need to prove that for $f$ taking values in $[0, 1]$ and $\mathbb{E}f \leq 1/2$:

$$ (*) \quad \mathbb{E} \frac{\varphi(h_t) \| H(h_t) \|_F^2}{(1 + \| \nabla h_t \|^2)^{3/2}} \geq c(\mathbb{E}f) \left( \mathbb{E} \| H(h_t) \|_F^2 \right)^4 $$

- For this, using the reverse log Sobolev inequality we prove that for $t$ large enough:

$$ (**) \quad \| \nabla h_t \| \leq 1 \text{ a.s.}, \quad \| \nabla f_t \| \leq \frac{\sqrt{2e^{-t}}}{\sqrt{1 - e^{-2t}}} f_t \sqrt{\log \frac{1}{f_t}} \text{ a.s} $$

- We then use (**), the concavity of $I$, the reverse-Hölder inequality, and reverse hyper-contractivity to show that:

$$ (***) \quad \mathbb{E} \left( \varphi(h_t) \| H(h_t) \|_F^2 \right) \geq cI^2 (\mathbb{E}f) (\mathbb{E} \| H(h_t) \|_F)^2 $$
The main challenge

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- We then use (**), the concavity of $I$, the reverse-Hölder inequality, and reverse hyper-contractivity to show that

\[ (***) \quad \mathbb{E}(\varphi(h_t)\|H(h_t)\|_F^2) \geq cI^2(\mathbb{E}f)(\mathbb{E}\|H(h_t)\|_F)^2 \]

- Finally using almost all of the tools before and additionally concentration of measure and Hanson-Wright inequalities we prove that for $t$ large enough

\[ (****) \quad \left(\mathbb{E}\|H(h_t)\|_F^3\right)^{1/3} \leq \sqrt{\log(1/(\mathbb{E}f))} \]
Combining the pieces

▶ By (***) for some $t_*$ and $t > t_*$:

$$
\mathbb{E} \frac{\varphi(h_t) \| H(h_t) \|_F^2}{(1 + \| \nabla h_t \|^2)^{3/2}} \geq c \mathbb{E} (\varphi(h_t) \| H(h_t) \|_F^2)
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Sadly - the square is outside the expectation.

However by Hölder's inequality \(\mathbb{E}(\|H(h_t)\|_F^2) \leq \left(\mathbb{E}\|H(h_t)\|_F\right)^{1/2} \left(\mathbb{E}\|H(h_t)\|_F^3\right)^{1/2}\). And therefore by (****) the upper bound on \(\mathbb{E}\|H(h_t)\|_F\) yields an upper bound on \(\mathbb{E}(\varphi(h_t)\|H(h_t)\|_F^2)\).
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- By (***) for $t > t_*$:

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\mathbb{E} (\| H(h_t) \|^2_F) \leq \left( \mathbb{E} \| H(h_t) \|_F \right)^{1/2} \left( \mathbb{E} \| H(h_t) \|^3_F \right)^{1/2}
\]

and therefore by (****) the upper bound on \( \mathbb{E} \| H(h_t) \|_F \) yields an upper bound on \( \mathbb{E} (\| H(h_t) \|^2_F) \).
Open Problems

- Prove that if \( f = 1_A \) satisfies \( \delta(f) \leq \delta \) then there exists a half space \( B \) such that \( \gamma_n(A \Delta B) \leq C\delta^{1/2} \).
Open Problems

- Prove that if $f = 1_A$ satisfies $\delta(f) \leq \delta$ then there exists a half space $B$ such that $\gamma_n(A \Delta B) \leq C\delta^{1/2}$.
- Analyze equality case and robustness of isoperimetric problems for other log-concave measures.
Borell showed that if $\gamma_n(B) = \gamma_n(A)$ and $B$ is a half-space then

$\mathbb{E}[1_A P_t 1_A] \leq \mathbb{E}[1_B P_t 1_B]$
Current and Future Work (M+Neeman)

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\end{align*}$

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Problem: Is there a robust version?

A’s: Yes, Yes (M + Neeman, 2012-3).