Optimal Gaussian Partitions with Application and Open Problems

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Optimal Gaussian Partitions

Optimal Gaussian Partitions

How to partition

- $\mathbb{R}^n$ (n is unbounded)
- into $r \times q$ parts $f_i^{-1}(a)$ for $1 \le i \le r$ and $1 \le a \le q$,
- of prescribed Gaussian measures $m_{i,a}$ with $\sum_a m_{i,a} = 1$,
- such that r Gaussian vectors $X_1, \ldots, X_r \in \mathbb{R}^n$ with prescribed covariance structure $Cov(X_i, X_j) = V_{i,j} I_n$
- maximize the expected value of "combinatorial quantity" depending only on $(f_i(X_i))_{i=1}^r$.

Notes

- An asymptotic geometric problem (dimension is unbounded).
- value increases with dimension, maximum is supremum.

Formal definition

Optimal Gaussian Partition

Given:

- $H : [q]^r \to \mathbb{R}$ (combinatorial weights)
- $m \in M_{r \times q}$ a stochastic matrix (parts sizes).
- $0 \le V \in M_{r \times r}$ with $V_{i,i} = 1$ for all i (covariance structure).

Define

$$M(H, m, V) := \sup \mathbb{E}[H(f_1(X_1), \ldots, f_r(X_r))]$$

where the sup is taken over all

- dimensions n,
- $f_i : \mathbb{R}^n \to [q]$ s.t.
- $\mathbb{P}[f_i(X) = a] = m_{i,a}$ for all $1 \le i \le r$ and $1 \le a \le q$.
- $X_1, \ldots, X_r \in \mathbb{R}^n$ are jointly Gaussian with $Cov[X_i, X_j] = V_{i,j} I_n$.

What's known? $q = 2$ parts with $r = 2$

Thm: (C. Borell 1985)

When $r = 2, q = 2$, general m and

$$H(a,b) = 1(a = b), \quad V = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \rho > 0$$

Maximum is obtained in dimension $n = 1$ and

$$f_i(x) = \begin{cases} 1 & x < t. \\ 2 & x \geq t. \end{cases}, \qquad P[X > t] = m_{i,2}.$$

In words

Partition of $\mathbb{R}^n$ into two parts of equal measure which maximizes the probability that two correlated Gaussians will fall in the same part is given by a half-space.

What's known? $q = 2$ parts with general r

Thm: (Isaksson-M 2011)

When $r \geq 2, q = 2,\ m = (m_1, m_2)$,
$H(a, b, c, \ldots) = 1(a = b = c = \ldots)$ and

$$V = \begin{pmatrix} 1 & \rho & \ldots & \rho \\ \rho & 1 & \rho \ldots & \\ \vdots & \ddots & \ddots & \ldots \end{pmatrix}, \rho > 0$$

Maximum is obtained in dimension $n = 1$ and

$$f_i(x) = \begin{cases} 1 & x < t. \\ 2 & x \geq t. \end{cases}, \qquad P[X > t] = m_{i,2}.$$

What else is known?

Nothing.

Proof Techniques

Borell's proof (1985)

Ehrhard symmetrization.

Isaksson-M approach (2011)

- Formulate a spherical statement.
- Prove Spherical Statement using Rearrangement Inequalities.
- Project to a small number of coordinates to obtain Gaussian results

Spherical Statement

Spherical Partition Problem

Given n, $\quad 0 \leq \Sigma \in \mathbb{R}^{k \times k}$, $\quad (m_1, \ldots, m_k) \in (0,1)^k$, Find $\sup P(X_1 \in A_1, \ldots, X_k \in A_k)$ where

- $X_1', \ldots, X_k'$ are jointly normal with $Cov(X_i', X_j') = \Sigma_{i,j} I_n$
- $X_i = \frac{X_i'}{\|X_i'\|_2}$
- sup is over A_i with $\mu(X_i \in A_i) = m_i$ where μ is the Haar measure on the $(n-1)$-sphere.

Thm: Optimal Spherical Partition

If $\Sigma_{i,j}^{-1} \leq 0$ for all $i \neq j$ then:

$$P(X_1 \in A_1, \ldots, X_k \in A_k) \leq P(X_1 \in H_1, \ldots, X_k \in H_k),$$

where $H_i = \{x : x_1 \leq a_1\}$ with $\mu(H_i) = \mu(A_i) = m_i$.

Optimal Spherical Partition - Proof Sketch

Express $P(X_1 \in A_1, \ldots, X_k \in A_k)$ in terms of independent normals $Z_i \sim N(0, c_i I_n)$. Writing $W_i = Z_i / \|Z_i\|_2$ to obtain

$$C_1 \mathbb{E}\left[1_{\{W_1 \in A_1, \ldots, W_k \in A_k\}} \prod_{1 \le i < j \le k} e^{-\left(\Sigma^{-1}\right)_{i,j} \langle Z_i, Z_j \rangle} \right] =$$

$$C_1 \mathbb{E}\left[1_{\{W_1 \in A_1, \ldots, W_k \in A_k\}} \prod_{1 \le i < j \le k} e^{-\left(\Sigma^{-1}\right)_{i,j} \langle W_i, W_j \rangle \|Z_i\|_2 \|Z_j\|_2} \right]$$

Conditioned on $\|Z_i\|_2$, W_i are uniformly distributed on the sphere and $\langle W_i, W_j \rangle$ decreases in $\|W_i - W_j\|$.
Therefore can apply extended Riesz Inequality (Burchard-01, Morpurgo-02) to conclude maximum is obtained for half-spaces H_i.

- Take $n \leq m \to \infty$.
- $X_i \in S^{m-1}$, $Y_i \in R^n$ with the same covariance structure Σ.
- $Z_i =$ first n coordinates of X_i.
- $\sqrt{m}(Z_1, \ldots, Z_k) \to_{m \to \infty} (Y_1, \ldots, Y_k)$ in distribution.
- Spherical bound implies Gaussian bound.
- Some approximation agruments needed when sets are not closed.

Open Problem 1 - Finite Dimensionality?

1. Finite dimensionality

Is the supremum $M(H, m, V)$ a maximum? Is it obtained in a finite dimension?

1.a Finite dimensionality variant

Same question assuming $f_s = f_1$ and $m_{s,j} = m_{1,j}$ for $1 \le s \le r$? (Conj. of O. Regev: $n = \infty$ for $r = 2, q = 2, H(a, b) = 1(a \neq b)$).

Comment : Approximate Finite Dimensionality

Find explicit $n(\epsilon, H)$ or $n(\epsilon, H, m, V)$ such that sup in dimension n is ϵ close to $M(H, m, V)$? (Seems doable using dimension reduction ideas (see Raghavendra-Steurer-09)).

Open Problem 2 - Other Optimal partitions?

More Examples

Find other optimal Gaussian partitions!

The Standard Simplex Conjecture (Isaksson-M-11)

Suppose $X, Y \sim N(0, I_n)$ and $Cov(X, Y) = \rho I_n$. Let $A_1, \ldots, A_q \subseteq \mathbb{R}^n$ be a partition of $\mathbb{R}^n$ and $S_1, \ldots, S_q \subseteq \mathbb{R}^n$ a standard simplex partition. Then,

i) If $\rho \geq 0$ and $A_1, \ldots, A_q$ is *balanced*, then

$$\mathbb{P}((X, Y) \in A_1^2 \cup \cdots \cup A_q^2) \leq \mathbb{P}((X, Y) \in S_1^2 \cup \cdots \cup S_q^2) \quad (1)$$

ii) If $\rho < 0$:

$$\mathbb{P}((X, Y) \in A_1^2 \cup \cdots \cup A_q^2) \geq \mathbb{P}((X, Y) \in S_1^2 \cup \cdots \cup S_q^2) \quad (2)$$

The Standard Simplex Partition

definition

For $n+1 \geq q \geq 2$, $A_1, \ldots, A_q$ is a *standard simplex partition* of $\mathbb{R}^n$ if for all i

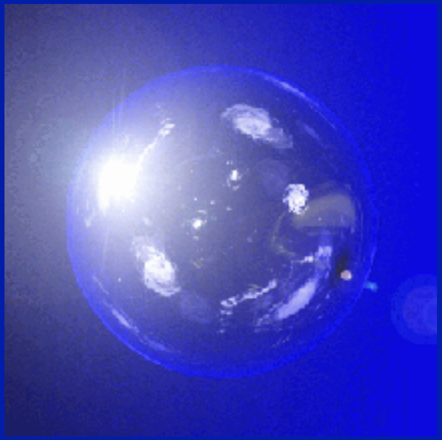
$$A_i \supseteq \{x \in \mathbb{R}^n | x \cdot a_i > x \cdot a_j, \forall j \neq i\} \qquad (3)$$

where $a_1, \ldots a_q \in \mathbb{R}^n$ are q vectors satisfying

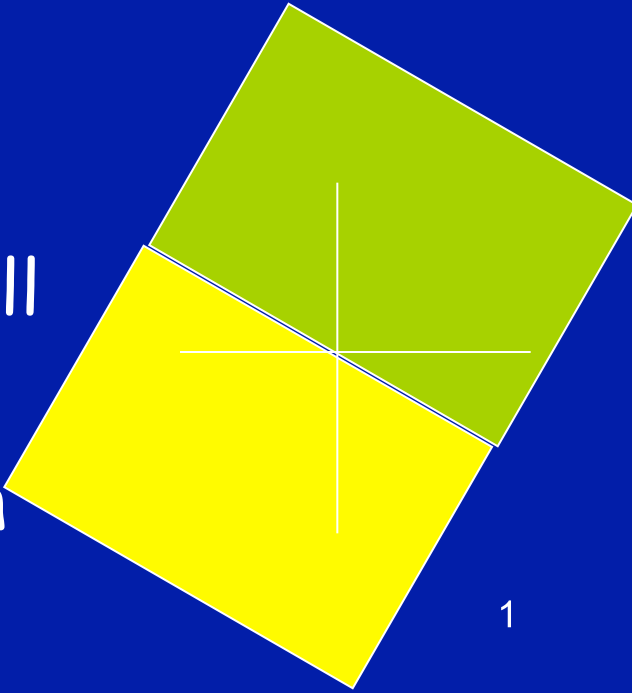
$$a_i \cdot a_j = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{q-1} & \text{if } i \neq j \end{cases} \qquad (4)$$

Isoperimetric context

- I. Ancient: Among all sets with $\nu_n(A) = 1$ the minimizer of $\nu_{n-1}(\partial A)$ is A = Ball.



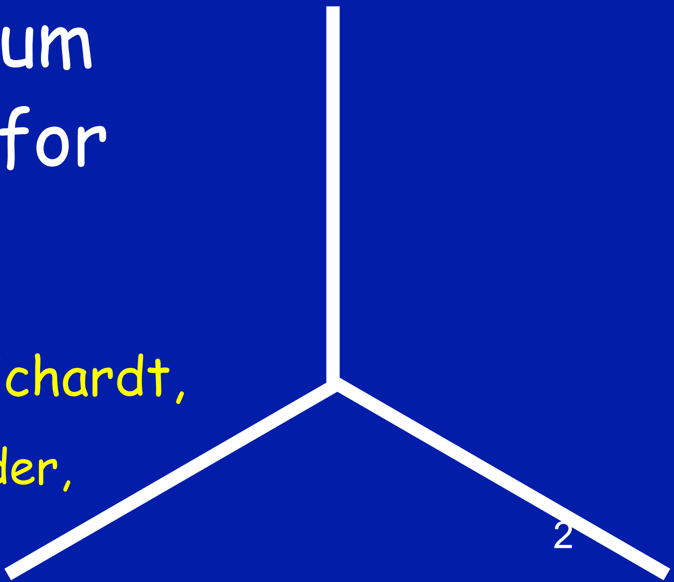
- II. Recent (Borell, Sudakov-Tsierlson 70's) Among all sets with $\gamma_n(A) = a$ the minimizer of $\gamma_{n-1}(\partial A)$ is A = Half-Space.
- III. More recent (Borell 85): For all ρ, among all sets with $\gamma(A) = a$ the maximizer of $E[A(N)A(M)]$ is given by A = Half-Space.



Double bubbles

- **Thm1** ("Double-Bubble"):
- Among all pairs of disjoint sets A,B with $\nu_n(A) = a\ \nu_n(B) = b$, the minimizer of $\nu_{\nu-1}(\partial A \cup \partial B)$ is a "Double Bubble"

- **Thm2** ("Peace Sign"):
- Among all partitions A,B,C of R^n with $\gamma(A) = \gamma(B) = \gamma(C) = 1/3$, the minimum of $\gamma(\partial A \cup \partial B \cup \partial C)$ is obtained for the "Peace Sign"



- 1. Hutchings, Morgan, Ritore, Ros. + Reichardt, Heilmann, Lai, Spielman 2. Corneli, Corwin, Hurder, Sesum, Xu, Adams, Dvais, Lee, Vissochi

Newer Isoperimetric Results

- <u>**Conj (Isaksson-M, Israel J. Math 2011):**</u>

  For all $0 \le \rho \le 1$:

  $\text{argmax } E[A(X)A(Y) + B(X)B(Y) + C(X)C(Y)]$

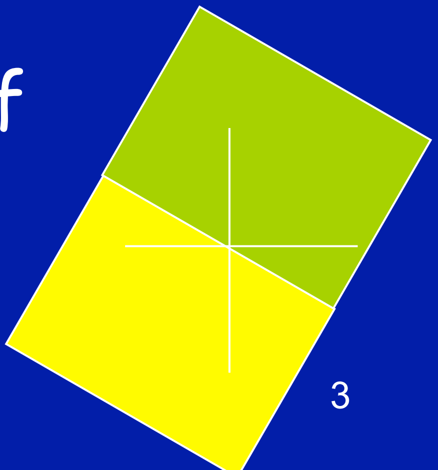
  = "Peace Sign"

  where max is over all partitions (A,B,C) of R^n with $\gamma_n(A) = \gamma_n(B) = \gamma_n(C) = 1/3$ is

Peace sign

Later we'll see applications

- <u>Challenges:</u>
- Can one extend the double bubble proof to the Gaussian setup?
- Develop symmetrisation techniques for partition into 3 parts.

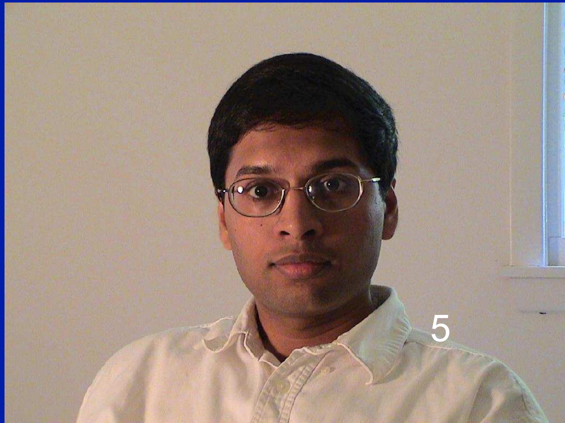


Motivation

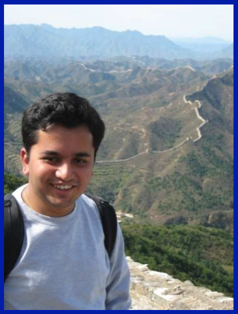
- Approximate Optimization
  - Unique Games and Optimization.
- Quantitative Social choice
  - Quantitative Arrow theorem.

Approximate Optimization

- Many optimization problems are NP-hard.
- Instead: Approximation algorithms
- These are algorithms that guarantee to give a solution which is at least
- α OPT or OPT - ε.
- S. Khot (2002) invented a new paradigm for analyzing approximation algorithms – called UGC (Unqiue Games Conjecture)



Other Approximation problems



- Work of KKMO04,MOO-05 gives best approximation factor for Max-Cut.
- Crucially uses Borell's optimal partition.
- A second result using Invariance of M 08;10
- Raghavendra 08: Duality between Algorithms and Hardness for Constraint Satisfaction Problems.
- $\Rightarrow$ Solution to Gaussian partition problem implies "best" approximation factor/ algorithm for the corresponding optimization problem.

Majority is Stablest

- Let $(X_i, Y_i) \in \{-1,1\}^n$ & $E[X_i] = E[Y_i] = 0$; $E[X_i\ Y_i] = \rho$.
- Let $Maj(x) = sgn(\sum x_i)$.
- <u>Thm (Sheffield 1899):</u>
- $E[Maj(X)\ Maj(Y)] \rightarrow M(\rho) := (2 \arcsin \rho)/\pi$
- <u>Thm (MOO; "Majority is Stablest"):</u>
- Let $f : \{-1,1\}^n \rightarrow \{-1,1\}$ with $E[f] = 0$.
- $I_i(f) := P[f(X_1,\ldots,X_i,\ldots,X_n) \neq f(X_1,\ldots,-X_i,\ldots,X_n)]$,
- $I = \max I_i(f)$
- Then: $E[f(X)\ f(Y)] \leq M(\rho) + C/\log^2(1/I)$
- Proof follows Borell's result and invariance.

Quantitative Social Choice

- Quantitative social choice studies different voting methods in a quantitative way.

- Standard assumption is of uniform voting probability.

- A "stress-test" distribution
  Bias distributions are not sensitive to errors/manipulation/paradoxes etc.

- Consider general voting rule
- $f: \{-1,1\}^n \rightarrow \{-1,1\}$ or $f : [q]^n \rightarrow [q]$ etc.

Errors in Voting

- Suppose each vote is re-randomized with probability ε (by voting machine):
- Majority is Stablest (MOO 05;10):
- Majority minimizes probability of error in outcome among low influence functions.
- Follows from Borll's partition result.

- Plurality is Stablest (IM) 11:
- The statement that
- Plurality minimizes probability of error in outcome among low influence functions is equivalent to
- Peace-Sign conjecture.

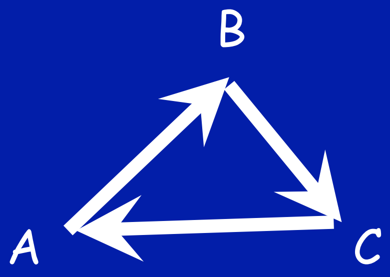
Errors in Voting

- Majority is Most Predictable (M 08; 10):
- Suppose each voter is in a poll with prob. p independently.
- Majority is most predictable from poll among all low influence functions.

- Next Example – Arrow theorem
- Fundamental theorem of modern social choice.

Condorcet Paradox

- n voters are to choose between 3 options / candidates.
- Voter i ranks the three candidates A, B & C via a permutation $\sigma_i \in S_3$
- Let $X^{AB}_i = +1$ if $\sigma_i(A) > \sigma_i(B)$
  $X^{AB}_i = -1$ if $\sigma_i(B) > \sigma_i(A)$
- Aggregate rankings via: $f,g,h : \{-1,1\}^n \to \{-1,1\}$.
- Thus: A is preferred over B if $f(x^{AB}) = 1$.
- A Condorcet Paradox occurs if:
  $f(x^{AB}) = g(x^{BC}) = h(x^{CA})$.
- Defined by Marquis de Condorcet in 18'th century.

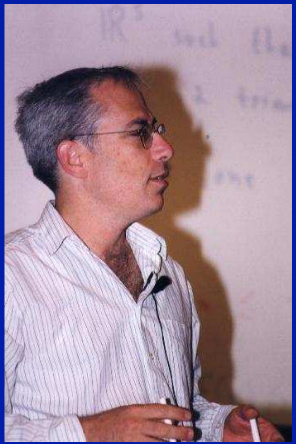


Arrow's Impossibility Thm

- Thm (Condorecet): If $n > 2$ and f is the majority function then there exists rankings $\sigma_1,\ldots,\sigma_n$ resulting in a Paradox

- Thm (Arrow's Impossibility): For all $n > 1$, unless f is the dictator function, there exist rankings $\sigma_1,\ldots,\sigma_n$ resulting in a paradox.
- Arrow received the Nobel prize (72)

Probability of a Paradox



- What is the probability of a paradox:
- $PDX(f) = P[f(x^{AB}) = f(x^{BC}) = f(x^{CA})]$?
- Arrow's: f = dictator iff $PDX(f) = 0$.

- Thm(Kalai 02): Majority is Stablest for $\rho=1/3 \Rightarrow$ majority minimizes probability of paradox among low influences functions (7-8%).

- Thm(Isacsson-M 11): Majority maximizes probability of a unique winner for any number of alternatives.
- (Proof uses invariance + Exchangble Gaussian Theorem)

Summary

- Prove the "Peace Sign Conjecture" (Isoperimetry)
- $\Rightarrow$ "Plurality is Stablest" (Low Inf Bounds)
- $\Rightarrow$ MAX-3-CUT hardness (CS) and voting.

$+ \Rightarrow$ New isoperimetric results.


Thank you!
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Lindeberg & Berry Esseen

- Let X_i = +/- w.p $\frac{1}{2}$, $N_i \sim N(0,1)$ ind.
- $f(x) = \sum_{i=1}^n c_i x_i$ with $\sum c_i^2 = 1$.
- Thm: (Berry Esseen CLT):
- $\sup_t |P[f(X) \le t] - P[f(N) \le t]| \le 3 \max |c_i|$
- Note that $f(N) = f(N_1,\ldots,N_n) \sim N(0,1)$.
- Lindeberg idea: can replace X_i with N_i as long as all coefficients are small.
- Q: can this be done for other functions f? e.g. polynomials?

Some Examples

- Q: Is it possible to apply Lindeberg principle to other functions with small coefficients?

- Ex 1: $f(x) = (n^3/6)^{-1/2} \sum_{i<j<k} x_i\, x_j\, x_k$ → Okay
- Limit is $N^3 - 3N$

- Ex 2: $f(x) = (2n)^{-1/2}\, (x_1 - x_2)\, (x_1 + \ldots + x_n)$ → Not OK
- For X: $P[f(X) = 0] \geq \frac{1}{2}$.

Invariance Principle

- Thm (MOO := M-O' Donnell-Oleszkiewicz; FOCS05, Ann. Math10):
- Let $Q(x) = \sum_S c_s X_s$ be a multi-linear polynomial of degree d with $\sum c_S^2 = 1$.
- $I_i(Q) := \sum_{S\,:\,i \in S} c_S^2 \qquad I(Q) = \max_i I_i(Q)$
- Then:
- $\sup_t |P[f(X) \leq t] - P[f(N) \leq t]| \leq 3\, d\, I^{1/8d}$
- Works if X has $2+\varepsilon$ moments + other setups.

The Role of Hyper-Contraction

- Pf Ideas:
- Lindeberg trick (replace one variable at a time)
- Hyper-contraction allows to bound high moments in term of lower ones.
- X is $(2, q > 2, a)$ Hyper-contractive if for all x:
- $|x + a X|_q \leq |x + X|_2$
- Key fact: A degree d polynomial of $(2,q,a)$ variables is $(2,q,a^d)$ hyper-contractive.
- Key fact 2: I f $|X|_q < \infty$ then it is $(2,q,a)$ hyper-contractive for $a = |X|_2/(q-1)^{1/2} |X|_q$

Related Work

- Many works generalizing Lindeberg idea:
- Chatterjee 06: Lindeberg - worst case influence.
- Rotar 79: Similar result no Berry Esseen bounds.
- New in our work: use of hyper-contraction.
- Classical results for U,V statistics.
- M (FOCS 08, Geom. and Functional Analysis 10):
- Multi-function versions.
- General “noise”.
- Bounds in terms of cross influences.

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- Thm (MOO; "Majority is Stablest"):
- Let $f : \{-1,1\}^n \to \{-1,1\}$ with $E[f] = 0$.
- $I_i(f) := P[f(X_1,\ldots,X_i,\ldots,X_n) \neq f(X_1,\ldots,-X_i,\ldots,X_n)]$,
- $I = \max I_i(f)$
- Then: $E[f(X)\ f(Y)] \leq M(\rho) + C/\log^2(1/I)$

Majority is Stablest – Pf Idea

- Pf Ideas: Use "non-linear invariance" +
- "noise truncation" (reduction to bdd degree f's) equivalent to the following regarding normal vectors:
- Let N,M be two n-dim normal vectors
- where (N_i,M_i) i.i.d. & $E[N_i] = E[N_i] = 0$; $E[N_i\ M_i] = \rho$.
- Then
- (*) Argmax $\{E[f(N)\ f(M)] : E[f] = 0, f \in \pm 1\}$ is $f(x) = sgn(x_1)$.
- (*) was proved by C. Borell 1985.

Majority is Stablest – Context

- Conext:
- Implies social choice conjecture by Kalai 2002.
- Proves the conjecture of Khot-Kindler-M-O'Donnell 2005 in the context of approximate optimization.
- Strengthen results of Bourgain 2001.
- More general versions proved in M-10
- M-10 allows truncation in general "noise" structure.
- E.g: In M-10: Majority is most predictable:
- Among low influence functions majority outcome is most predictable give a random sample of inputs.

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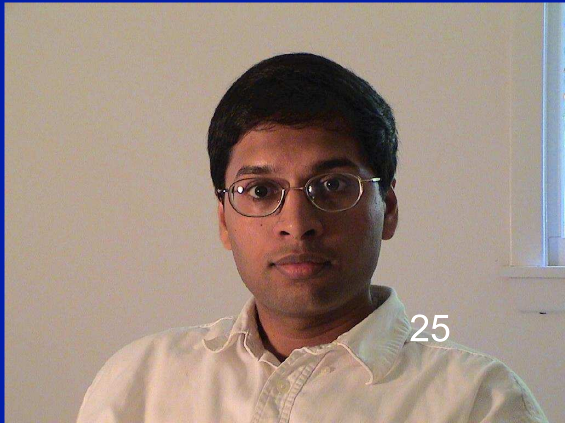
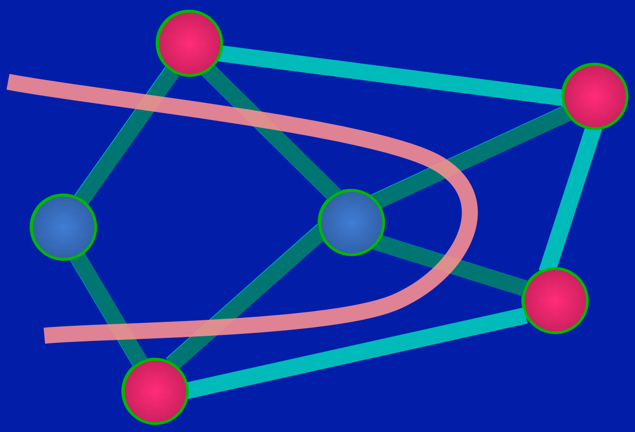


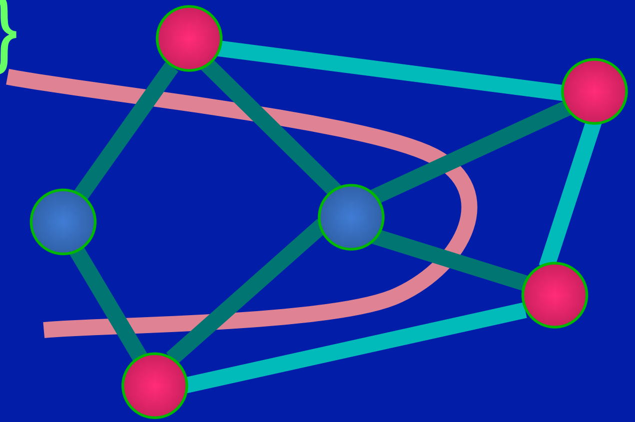
Example 1: The MAX-CUT Problem

- $G = (V,E)$
- $C = (S^c,S)$, partition of V
- $w(C) = |(S \times S^c) \cap E|$
- $w : E \longrightarrow R^+$
- $w(C) = \sum_{e \in E \cap S \times S^c} w(e)$



Example: The Max-Cut Problem

- OPT = OPT(G) = $\max_c \{|C|\}$



- **MAX-CUT problem:**
  find C with w(C)= OPT

- **α-approximation:**
  find C with w(C) $\geq \alpha \cdot$OPT

- Goemans-Williamson-95:
- Rounding of
- Semi-Definite Program gives an

α = .878567 approximation algorithm.

MAX-Cut Approximation

- Thm (KKMO = Khot-Kindler-M-O' Donell, FOCS 2004, Siam J. Computing 2007):
- Under UGC, the problem of finding an $\alpha > a_{GW} = 0.87\ldots$ approximation for MAX-CUT is NP-hard.
- Moral: Semi-definite program does the best.

- Thm (IM-2010): Same result for MAX-q-CUT assuming the Peace-Sign Conjecture.

Other Approximation problems

- Work of KKMO04,MOO-05 show gives best approximation factor for Max-Cut.
- Crucially uses Borell's optimal partition.
- A second result using Invariance of M 08;10
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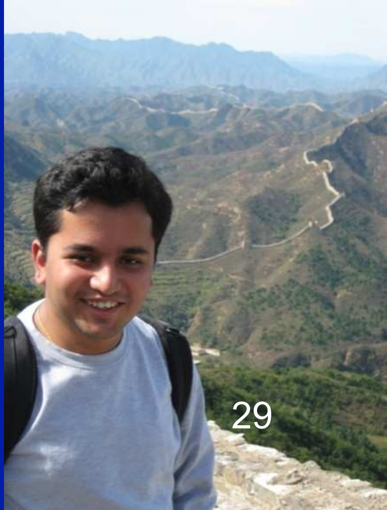
- $\Rightarrow$ Any optimal solution to
- Gaussian partition problem gives "best" approximation factor/algorithm for the corresponding optimization problem.



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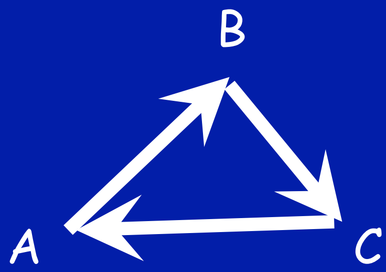
- A Condorcet Paradox occurs if:
  $f(x^{AB}) = g(x^{BC}) = h(x^{CA})$.
- Defined by Marquis de Condorcet in 18' th century.

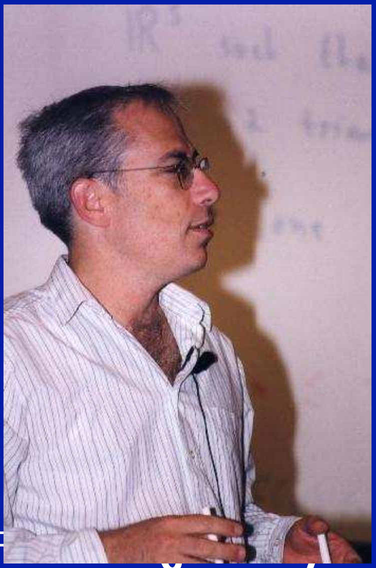


Arrow's Impossibility Thm

- Thm (Condorecet): If $n > 2$ and f is the majority function then there exists rankings $\sigma_1,\dots,\sigma_n$ resulting in a Paradox

- Thm (Arrow's Impossibility): For all $n > 1$, unless f is the dictator function, there exist rankings $\sigma_1,\dots,\sigma_n$ resulting in a paradox.
- Arrow received the Nobel prize (72)

Probability of a Paradox



- What is the probability of a paradox:
- $PDX(f) = P[f(x^{AB}) = f(x^{BC}) = f(x^{CA})]$?
- Arrow's: f = dictator iff $PDX(f) = 0$.
- Thm(Kalai 02): Borell's optimal partition = is Stablest for $\rho=1/3 \rightarrow$ majority minimizes probability of paradox among low influences functions (7-8%).
- Thm(Isacsson-M 11): Majority maximizes probability of a unique winner for any number of alternatives.
- (Proof uses invariance + Exchangble Gaussian Theorem)

Probability of a Paradox

- Arrow's: f = dictator iff $PDX(f) = 0$.
- Kalai 02: Is it true that $\forall\ \varepsilon\ \exists\ \delta$ such that
- if $PDX(f) < \delta$
- then f is ε close to dictator?
- Kalai 02: Yes if there are 3 alternatives under technical condition.
- M-11: True for any number of alternatives.
- Pf uses Majority is stablest and inverse hyper-contractive inequalities.

Summary

- Prove the "Peace Sign Conjecture" (Isoperimetry)
- $\Rightarrow$ "Plurality is Stablest" (Low Inf Bounds)
- $\Rightarrow$ MAX-3-CUT hardness (CS) and voting.

+ $\Rightarrow$ Results in Geometry.


Thank you!
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