The Geometry of Manipulation - a Quantitative Proof of the Gibbard Satterthwaite Theorem

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Truthfulness in Voting

- <u>Question:</u> Which voting methods have the property that:

- voting truthfully is a dominant strategy?

Truthfulness in Binary Voting

- <u>Question:</u> Which voting methods have the property that:

- voting truthfully is a dominant strategy?

- voters always vote according to true preference?

Example: FOCS 2050?

- Assume Plurality vote with the following preferences:

Beijing

Zurich

Geneva

B	B	Z	G
Z	G	G	Z
G	Z	B	B
25%	20%	35%	20%

Choice Functions and Manipulation

Definition: A social choice function F associates to each collection of n rankings a winner:

$F : S(A,B,\ldots,K)^n \rightarrow \{A,B,C,D,\ldots,K\}$

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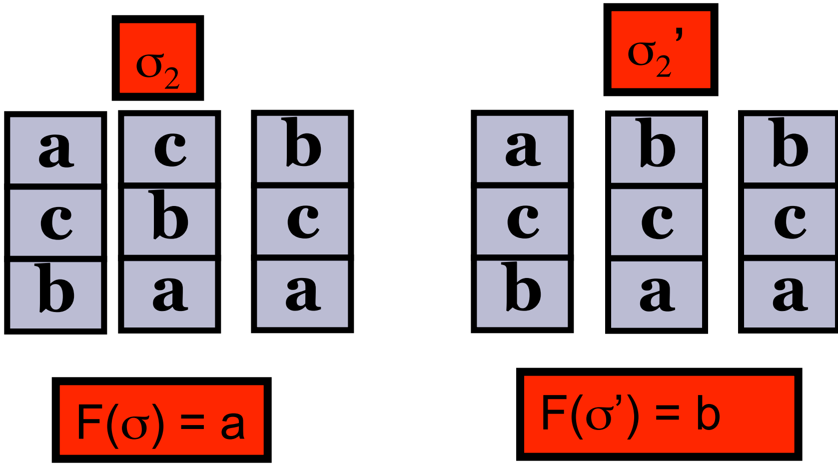
$\sigma_i(F(\sigma')) > \sigma_i(F(\sigma))$



Example: Manipulation by voter 2

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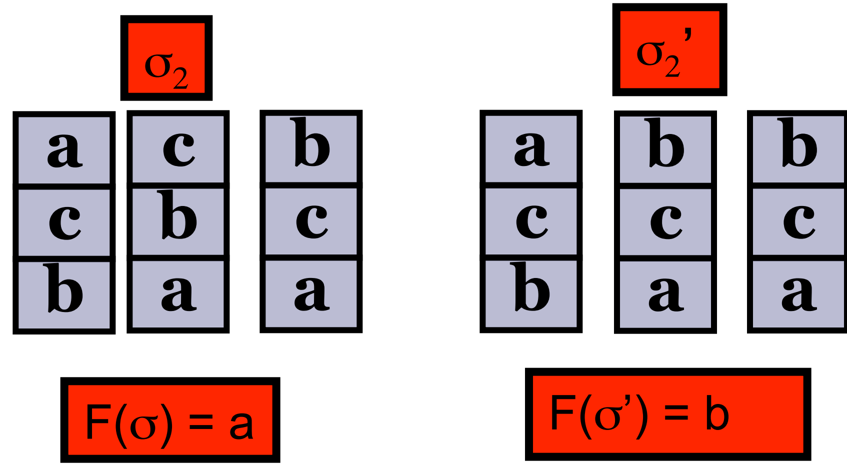
- F is strategy proof if there is no voter that can manipulate it.



Example: Manipulation by voter 2

Gibbard–Satterthwaite Thm

- Thm (Gibbard-Satterthwaite 73,75):

  If F ranks $k \geq 3$ alternatives,
- is onto / neutral
- strategy proof

Then F is a dictator

- Neutral := "all alternatives are treated equaly"

The GS Theorem – Computational and Quantitative Aspects

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The GS Theorem – Computational and Quantitative Aspects

- Q1: Perhaps manipulating is computationally hard?
- Q2: Perhaps for most voting profiles it is impossible to manipulate (assuming uniform measure).
- Def: $M(F) = P[\sigma$: some voter can manip F at $\sigma]$.
- Notation: Write $D(F,G) = P(F(\sigma) \neq G(\sigma))$.
  $D(F,D_k(n)) = \min \{ D(F,G) : G \text{ a dictator}\}$

Comp & Qaunt. Aspects

- Bartholdi, Orlin (91), Bartholdi,Tovey Trick (93):

  Manipulation for a voter for some voting schemes is NP hard (for large # of alternatives k).

- Sandholm, Conitzer (93, 95) etc. : Hard on average?

- Conj (Friedgut-Kalai-Nisan 08): Random manipulation gives $M(F) \geq \mathrm{poly}(n^{-1}, k^{-1}, D(F, D_k(n)))$.

- Thm (FKN 08): For $k=3$ alternatives, and neutral F, it holds that $M(F) \geq c\, n^{-1} D(F,D_k(n))^2$

  (uniform measure, no computational consequences)

- Xia & Conitzer 09 (many conditions, no k depenendcy), Dobzinski and Procaccia: (2 voters)

High Probability Manipulation

- Thm Isaksson-Kindler-M-10:

- If F is neutral and $k \geq 3$ then
- $M(F) \geq c\, n^{-3}\, k^{-30}\, D(F, D_k(n))^2$

- Moral: Proves FKN conj: Only functions that are close to strategy proof are the ones close to dictators.

High Probability Manipulation

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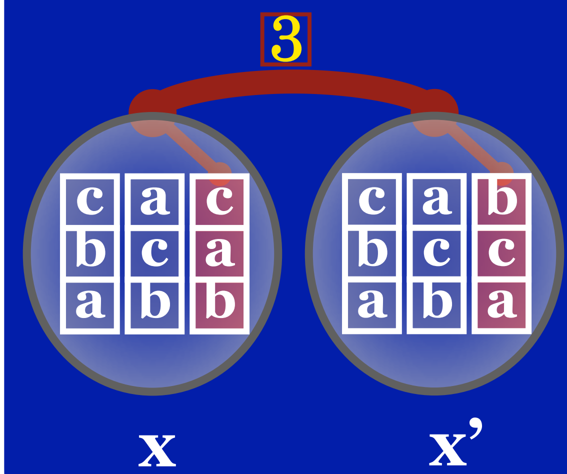
- Moreover: a simple randomized algorithm manipulates with probability at least $c\, n^{-3}\, k^{-30}\, D(F,D_k(n))^2$.

Comments

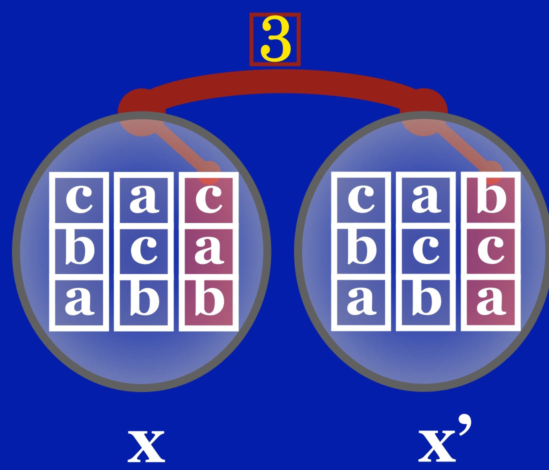
- Thm Isaksson-Kindler-M-10:
- If F is neutral then $M(F) \geq c\, n^{-3} k^{-10} D(F, D_k(n))^2$
- Moreover: An easy randomized algorithm manipulates with probability at least $c\, n^{-3} k^{-10} D(F, D_k(n))^2$.
- Note: For F = plurality on 3 alternatives and large # of voters n, manipulation exists only when two candidates are tied up. So $M(F) = O(n^{-1/2})$
- To the proof …

The rankings graph

- We consider the graph with vertex set $S(A,B,\ldots K)^n$
- $e=[x,x']$ is an edge on voter i, if $x(j) = x'(j)$ for $j \neq i$ and $x(i) \neq x'(i)$.
- For $F : S(A,\ldots K)^n \rightarrow \{A,\ldots,K\}$, we call $e=[x,x']$ a boundary edge if $F(x) \neq F(x')$.



[x,x'] is an edge on voter 3

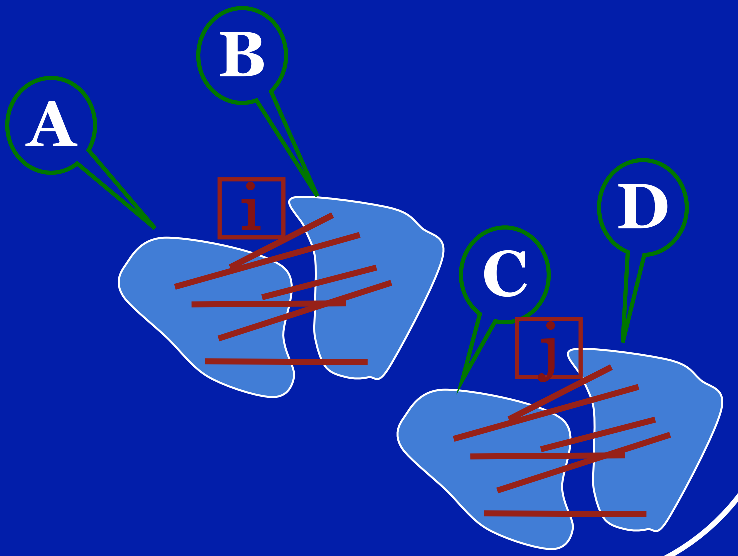


If F(x) = c and F(x') = a then [x,x'] is a boundary edge

Write:
$e \in \partial_3[c,a]$

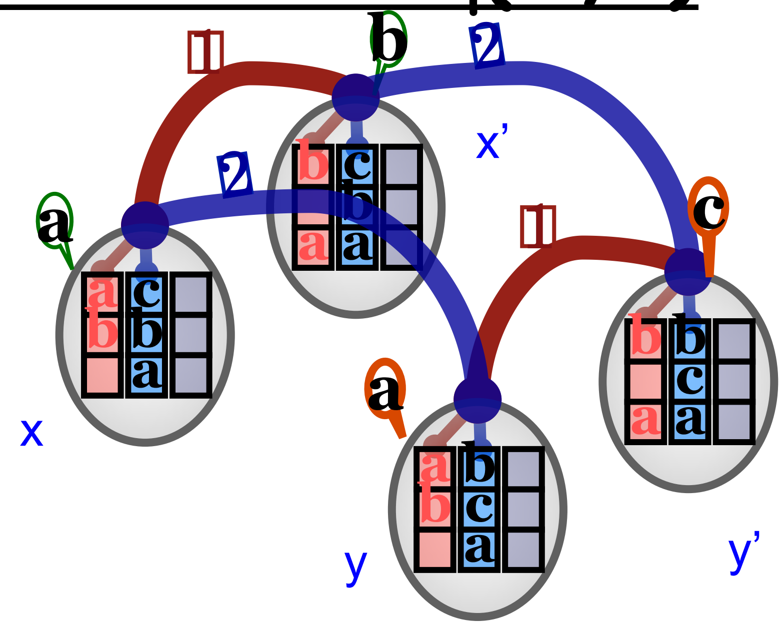
Boundaries

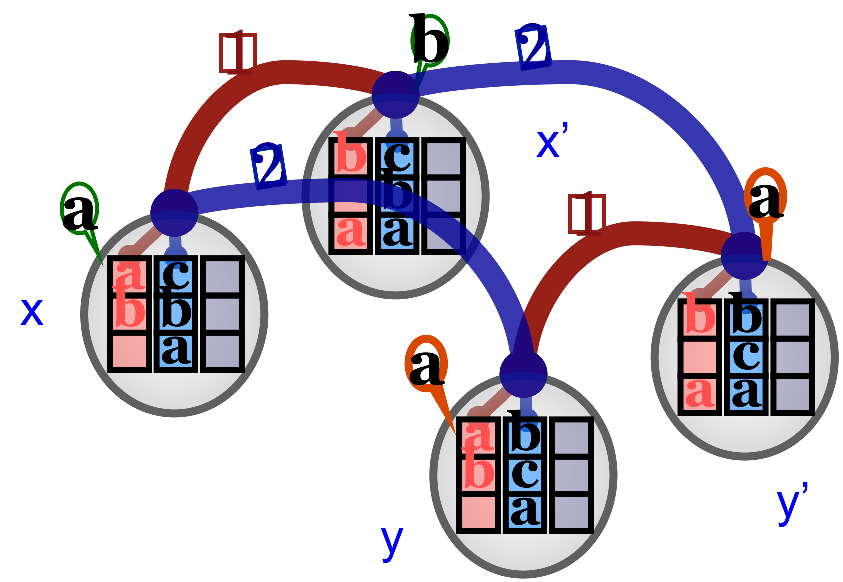
- Assume 4 alternatives, unif. distribution.
- An Isoperimetric Lemma:
- If F is ε far from all dictators and Neutral
- Then there exists voters $i \neq j$ and alternatives A,B,C,D s.t: $P[e \in \partial_i[A,B]] \geq \varepsilon\,(6n)^{-2}$,
  $P[e \in \partial_j[C,D]] \geq \varepsilon\,(6n)^{-2}$

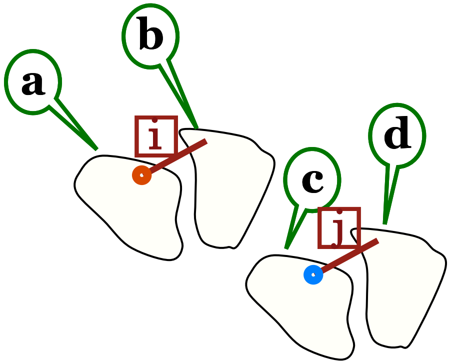


Main Idea: Paths and Flows on $\partial_i(A,B)$

- Key Property: The space $\partial_i[A,B]$ is “nice”:
- One can define “flows” and “paths” on it.
- &: ∂ $\partial_i[A,B]$ “=“ Manipulation points.
- Moves := changing voters rankings while preserving A,B order.





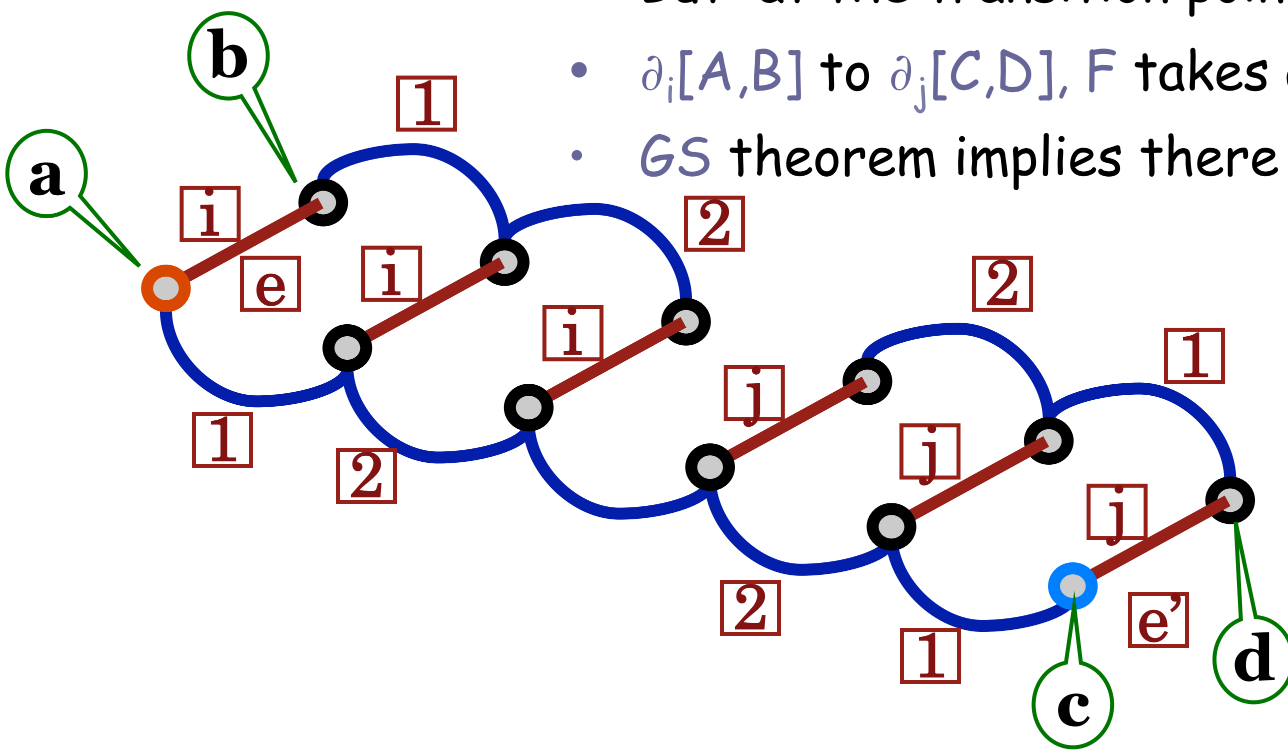


Using Canonical paths

Define a canonical path $\Gamma\{e,e'\}$ for all $e \in \partial_i[A,B]$ and $e' \in \partial[C,D]$ such that:

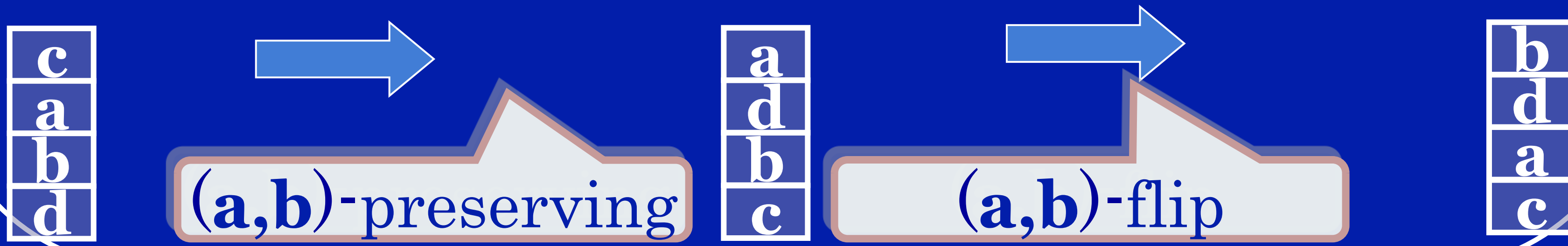
- The path begins at e and ends at e' and
- Path stays in $\partial_i[A,B] \cup \partial_j[C,D]$

  or encounters manipulation
- But: at the transition point m from
- $\partial_i[A,B]$ to $\partial_j[C,D]$, F takes at least 3 values so
- GS theorem implies there exists manipulation.



of Manipulation Points

- $P[M(F))] \geq (4!)^n R^{-1} P[\partial_i[A,B]] \times P[\partial_j[C,D]]$, where
- $R := \max_m \ \#\{ \{e,e'\} : m$ is manipulation for $\Gamma\{e,e'\} \}$
- Since: $|M(F)| \geq R^{-1} |\partial_i[A,B]| \times |\partial_j[C,D]|$
- Need to "decode" $\leq$ poly(k,n) $(4!)^n$ (e,e') from m.
- <u>Path to use:</u>
- 1. For all $1 \leq k \leq n$ make k'th coordinate agree with e' except A,B order agrees with e.
- 2. For all $1 \leq k \leq n$ flip (A,B) if need to agree e'.



of Manipulation Points

- Decoding:
- If $e=[x,x']$ and $e'=[y,y']$ suffices to decode (x,y) from m ($(k!)^2$ "pay" to know x' and y').
- Given a hint of size $4n$ know step of the path.
- Suffices for each coordinate s: given m_s decode at most $4!$ Options for (x_s, y_s).
- Given m_s either know x_s, or y_s or $4!/2$ options for x_s and 2 options for y_s.
- Decoding works!

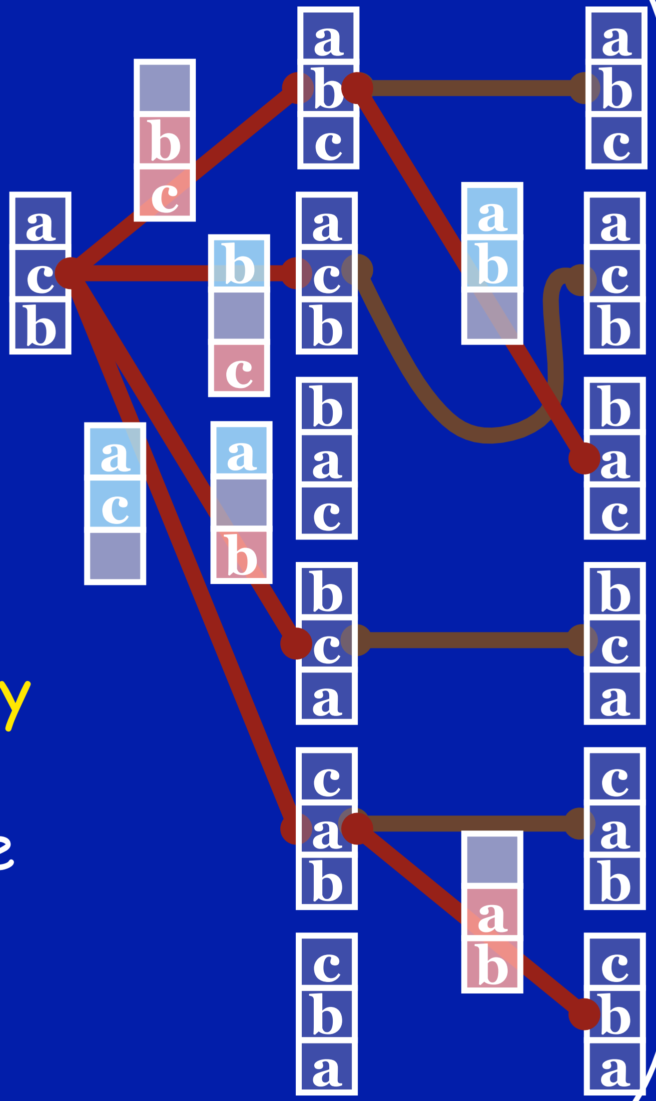
- So $P[M(F))] \geq (4!)^n R^{-1} P[\partial_i(a,b)] \times P[\partial_j(c,d)]$, "gives"
- $P[M(f)] \geq \varepsilon^2 (6n)^{-5}$.
- QED.

However …

- In fact, cheating in various places … - most importantly:
- Manipulation point = x or y up to 3 coordinates, so:
- $R \leq 2\ n\ 4^n\ (k!)^3$
- $P[M(f)] \geq (k!)^{-3}\ \varepsilon^2\ (6n)^{-5}$
- Fine for constant # of alternatives k, but not for large k.

+ Idea : Geometries on the ranking cubes

- To get polynomial dependency on k, use refined geometry:
- (x,x') ∈ Edges if x,x' differ in a single voter and an adjacent transposition.
- For a single voter:
- refined geometry = adjacent transposition card-shuffling.
- Prove: geometry = refined geometry up to poly. factors in k (spectral, isoperimetric quantities behave the same; Aldous-Diaconis, Wilson).
- Prove: Combinatorics still works.
- Gives manipulation by adj. transposition.



Open Problems

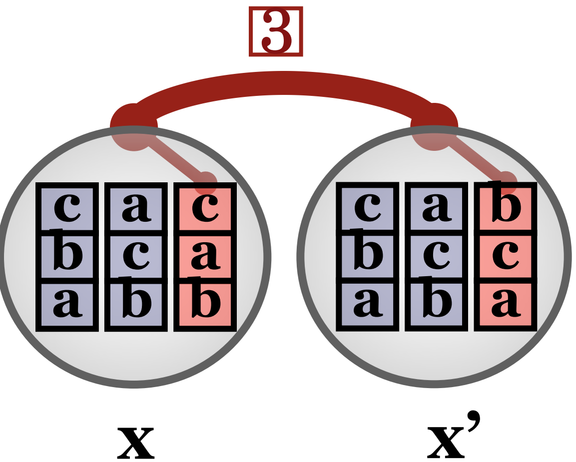
- Are there other combinatorial problems where high order interfaces play an interesting role?
- Can other isoperimetric tools be extended to higher order interfaces?
- Tighter results for GS theorem? Remove Neutrality?
- Proof without neutrality.

Brief summary

- If you haven't noticed it is impossible to avoid manipulation.
- You probably haven't noticed but it's possible to prove isoperimetric inequalities involving meetings of 3 bodies (not just 2!).

- Thank you for your attention!

3 Types of Boundary edges



F(x) = a F(x') = b

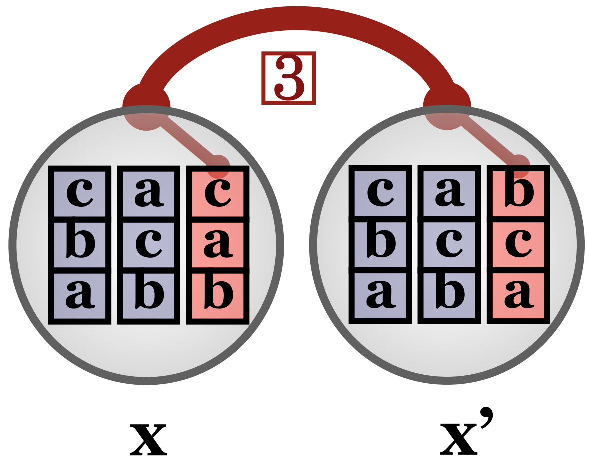
This edge is

monotone

and non-manipulable:

x ranks a above b

x' ranks b above a



F(x) = a F(x') = c

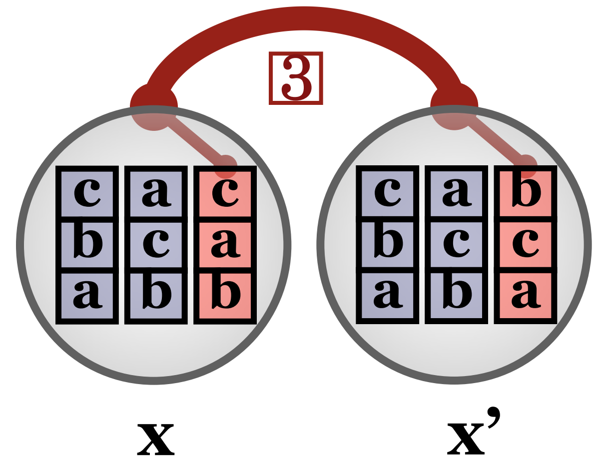
This edge is

monotone-neutral

and manipulable:

same order of

a,c in x,x'



F(x) = b F(x') = c

This edge is

anti-monotone

and manipulable:

x ranks c above b

x' ranks b above c