Stat260/CS294: Spectral Graph Methods **Lecture 16 - 03/17/2015**

Lecture: Modeling graphs with electrical networks

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Warning: these notes are still very rough. They provide more details on what we discussed in class, but there may still be some errors, incomplete/imprecise statements, etc. in them.

16 Electrical network approach to graphs

So far, we have been adopting the usual approach to spectral graph theory: understand graphs via the eigenvectors and eigenvalues of associated matrices. For example, given a graph $G = (V, E)$, we defined an adjacency matrix A and considered the eigensystem $Av = \lambda v$, and we also defined the Laplacian matrix $L = D - A$ and considered the Laplacian quadratic form $x^T Lx - \sum_{(ij)\in E}(x_i - x_j)^2$. There are other ways to think about spectral graph methods that, while related, are different in important ways. In particular, one can draw from *physical intuition* and define physical-based models from the graph G, and one can also consider more directly vectors that are obtained from various *diffusions and random walks* on G. We will do the former today, and we will do the latter next time.

16.1 A physical model for a graph

In many physical systems, one has the idea that there is an equilibrium state and that the system goes back to that equilibrium state when disturbed. When the system is very near equilibrium, the force pushing it back to the equilibrium state is quadratic in the displacement from equilibrium, one can often define a potential energy that in linear in the displacement from equilibrium, and then the equilibrium state is the minimum of that potential energy function.

In this context, let's think about the edges of a graph $G = (V, E)$ as physical "springs," in which case the weights on the edges correspond to a spring constant k. Then, the force, as a function of the displacement x from equilibrium, is $F(x) = kx$, and the corresponding potential energy is $U(x) = \frac{1}{2}kx^2$. In this case, i.e., if the graph is viewed as a spring network, then if we nail down some of the vertices and then let the rest settle to an equilibrium position, then we are interested in finding the minimum of the potential energy

$$\sum_{(ij)\in E} (x_i - x_j)^2 = x^T Lx,$$

subject to the constraints on the nodes we have nailed down. In this case, the energy is minimized when the non-fixed vertices have values equal to

$$x_i = \frac{1}{d_i} \sum_{(ij)\in E} x_j,$$

i.e., when the value on any node equals is the average of the values on its neighbors. (This is the so-called *harmonic property* which is very important, e.g., in harmonic analysis.)

As we have mentioned previously and will go into in more detail below, eigenvectors can be unstable things, and having some physical intuition can only help; so let's go a little deeper into these connections.

First, recall that the *standard/weighted geodesic graph metric* defines a distance $d(a, b)$ between vertices a and b as the length of the minimum-length path, i.e., number of edges or the sum of weights over edges, on the minimum-length path connecting a and b. (This is the "usual" notion of distance/metric on the nodes of a graph, but it will be different than distances/metrics implied by spectral methods and by what we will discuss today.)

Here, we will *model a graph $G = (V, E)$ as an electrical circuit.* (By this, we mean a circuit that arises in electromagnetism and electrical engineering.) This will allow us to use physical analogues, and it will allow us to get more robust proofs for several results. In addition, it allow us to define another notion of distance that is closer to diffusions.

As background, here are some physical facts from electromagnetism that we would like to mimic and that we would like our model to incorporate.

- A basic *direct current electrical circuit* consists of a *battery* and one or more *circuit elements* connected by wires. Although there are other circuit elements that are possible, here we will only consider the use of resistors. A battery consists of two distinct vertices, call them $\{a, b\}$, one of which is the source, the other of which is the sink. (Although we use the same terms, "source" and "sink," as we used with flow-based methods, the sources and since here will obey different rules.) A *resistor* between two points a and b, i.e., between two nodes in G, has an associated (undirected and symmetric) quantity r_{ab} called a *resistance* (and an associated conductance $c_{ab} = \frac{1}{r_{ab}}$). Also, there is a *current* Y_{ab} and a *potential difference* V_{ab} between nodes a and b.

Initially, we can define the resistance between two nodes that are connected by an edge to depend (typically inversely) on the weight of that edge, but we want to extend the idea of resistance to a resistance between any two nodes. To do so, an important notion is that of *effective resistance*, which is the following. Given a collection of resistors between nodes a and b, they can be replaced with a single effective resistor with some other resistance. Here is how the value of that effective resistance is determined.

- If a and b have a node c between them, i.e., the resistors are in series, and there are resistances $r_1 = R_{ac}$ and $r_2 = R_{cb}$, then the effective resistance between a and b is given by $R_{ab} = r_1 + r_2$.
- If a and b have no nodes between them but they are connected by two edges with resistances r_1 and r_2, i.e., the resistors are in parallel, then the effective resistance between a and b is given by $R_{ab} = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2}}$.
- These rules can be applied recursively.

From this it should be clear that the number of paths as well as their lengths contribute to the effective resistance. In particular, having k parallel edges/paths leads to an effective resistance that is decreased by $\frac{1}{k}$; and adding the first additional edge between two nodes has a big impact on the

effective resistance, but subsequent edges have less of an effect. Note that this is vaguely similar to the way diffusions and random walks behave, and distances/metrics they might imply, as opposed to geodesic paths/distances defined above, but there is no formal connection (yet!).

Let a voltage source be connected between vertices a and b, and let $Y > 0$ be the net current out of source a and into course b. Here we define two basic rules that our resistor networks must obey.

Definition 1. *The* Kirchhoff current law *states that the current Y_{ij} between vertices i and j (where $Y_{ij} = -Y_{ji}$) satisfies*

$$\sum_{j \in N(i)} Y_{ij} = \begin{cases} Y & i = a \\ -Y & i = b \\ \textit{free} & \textit{otherwise} \end{cases},$$

where $N(i)$ refers to the nodes that are neighbors of node i.

Definition 2. *The* Kirchhoff circuit/potential law *states that for every cycle C in the network,*

$$\sum_{(ij) \in C} Y_{ij} R_{ij} = 0.$$

From Definition 2, it follows that there is a so-called *potential function* on the vertices/nodes of the graph. This is known as Ohm's Law.

Definition 3. Ohm's Law *states that, to any vertex i in the vertex set of G, there is an associated potential, call it V_i, such that for all edges $(ij) \in E$ in the graph*

$$Y_{ij} R_{ij} = V_i - V_j.$$

Given this potential function, we can define the effective resistance between any two nodes in G, i.e., between two nodes that are not necessarily connected by an edge.

Definition 4. *Given two nodes, a and b, in G, the effective resistance is $R_{ab} = \frac{V_a - V_b}{Y}$.*

Fact. Given a graph G with edge resistances R_{ij}, and given some source-sink pair (a, b), the effective resistance exists, it is unique, and (although we have defined it in terms of a current) it does *not* depend on the net current.

16.2 Some properties of resistor networks

Although we have started with this physical motivation, there is a close connection between resistor networks and what we have been discussing so far this semester.

To see this, let's start with the following definition, which is a special case of the Moore-Penrose pseudoinverse.

Definition 5. *The* Laplacian pseudoinverse *is the unique matrix satisfying:*

1. *$L^+ \vec{1} = 0$; and*
2. *For all $w \perp \vec{1}$: $\quad L^+ w = v$ s.t. $Lv = w$ and $v \perp \vec{1}$.*

Given this, we have the following theorem. Note that here we take the resistances on edges to be the inverse of the weights on those edges, which is probably the most common choice.

Theorem 1. *Assume that the resistances of the edges of $G = (V, E)$ are given by $R_{ij} = \frac{1}{w_{ij}}$. Then, the effective resistance between any two nodes a and b is given by:*

$$\begin{aligned} R_{ab} &= (e_a - e_b)^T L^+ (e_a - e_b) \\ &= L^+_{aa} - 2L^+_{ab} + L^+_{bb}. \end{aligned}$$

Proof. The idea of the proof is that, given a graph, edge resistances, and net current, there always exists currents Y and potentials V satisfying Kirchhoff's current and potential laws; in addition, the vector of potentials is unique up to a constant, and the currents are unique. I'll omit the details of this since it is part of HW2. □

Since the effective resistance between any two nodes is well-defined, we can define the total effective resistance of the graph. (This is sometimes called the Kirchhoff index.)

Definition 6. *The* total effective resistance *is $R^{tot} = \sum_{ij=1}^{n} R_{ij}$.*

Before proceeding, think for a minute about why one might be interested in such a thing. Below, we will show that the effective resistance is a distance; and so the total effective resistance is the sum of the distances between all pairs of points in the metric space. Informally, this can be used to measure the total "size" or "capacity" of a graph. We used a similar thing (but for the geodesic distance) when we showed that expander graphs had a $\Theta(\log(n))$ duality gap. In that case, we did this, essentially, by exploiting the fact that there was a lot of flow to route and since most pairs of nodes were distance $\Theta(\log(n))$ apart in the geodesic distance.

The quantity R^{tot} can be expressed exactly in terms of the Laplacian eigenvalues (all of them, and not just the first one or first few). Here is the theorem (that we won't prove).

Theorem 2. *Let λ_i be the Laplacian eigenvalues. Then, $R^{tot} = n \sum_{i=1}^{n} \frac{1}{\lambda_i}$.*

Of course, we can get a (weak) bound on R^{tot} using just the leading nontrivial Laplacian eigenvalue.

Corollary 1.

$$\frac{n}{\lambda_2} \leq R^{tot} \leq \frac{n(n-1)}{\lambda_2}$$

Next, we show that the effective resistance is a distance function. For this reason, it is sometimes called the resistance distance.

Theorem 3. *The effective resistance R is a metric.*

Proof. We will establish the three properties of a metric.

First, from the above theorem, $R_{ij} = 0 \iff i = j$. The reason for this is since $e_i - e_j$ is in the null space of L^+ (which is the span($\vec{1}$)) iff $i = j$. Since the pseudoinverse of L has eigenvalues $0, \lambda_2^{-1}, \ldots, \lambda_n^{-1}$, it is PSD, and so $R_{ij} \geq 0$.

Second, since the pseudoinverse is symmetric, we have that $R_{ij} = R_{ji}$.

So, the only nontrivial thing is to show the triangle inequality holds.

To do so, we show two claims.

Claim 1. *Let* Y_{ab} *be the vector* $e_a - e_b = \begin{cases} 1 & \text{at } a \\ -1 & \text{at } b \\ 0 & \text{elsewhere,} \end{cases}$ *and let* $V_{ab} = L^+ Y_{ab}$. *Then,* $V_{ab}(a) \geq V_{ab}(c) \geq V_{ab}(b)$, *for all* c.

Proof. Recall that V_{ab} is the induced potential when we have 1 Amp going in a and 1 Amp coming out of b. For every vertex c, other than a and b, the total flow is 0, which means $\sum_{x \sim c} \frac{1}{R_{xc}}(V_{ab}(x) - V_{ab}(c)) = 0$, and it is easy to see $V_{ab}(c) = \frac{\sum_{x \sim c} C_{xc} V_{ab}(x)}{\sum_{x \sim c} C_{xc}}$ where $C_{xc} = \frac{1}{R_{xc}}$ is the conductance between x, c. $V_{ab}(c)$ has a value equal to the weighted average of values of $V_{ab}(x)$ at its neighbors. We can use this to prove the claim by contradiction. Assume that there exists a c s.t. $V_{ab}(c) > V_{ab}(a)$. If there are several such nodes, then let c be the node s.t. $V_{ab}(c)$ is the largest. In this case, $V_{ab}(c)$ is larger than the values at its neighbors. This is a contradiction, since V_c is a weighted average of the potentials at its neighbors. The proof of the other half of the claim is similar. (also $V_{ab}(a) \geq V_{ab}(b)$ as $V_{ab}(a) - V_{ab}(b) = R_{ab} \geq 0$) □

Claim 2. $R_{eff}(a, b) + R_{eff}(b, c) \geq R_{eff}(a, c)$

Proof. Let Y_{ab} and Y_{bc} be the external current from sending one unit of current from $a \to b$ and $b \to c$, respectively. Note that $Y_{ac} = Y_{ab} + Y_{bc}$. Define the voltages $V_{ab} = L^+ Y_{ab}$, $V_{bc} = L^+ Y_{bc}$, and $V_{ac} = L^+ Y_{ac}$. By linearity, $V_{ac} = V_{ab} + V_{bc}$. Thus, it follows that

$$R_{eff}(a, c) = Y_{ac}^T V_{ac} = Y_{ac}^T V_{ab} + Y_{ac}^T V_{bc}.$$

By Claim 1, it follows that

$$Y_{ac}^T V_{ab} = V_{ab}(a) - V_{ab}(c) \leq V_{ab}(a) - V_{ab}(b) = R_{eff}(a, b).$$

Similarly, $Y_{ac}^T V_{bc} \leq R_{eff}(b, c)$. This establishes the claim. □

The theorem follows from these two claims. □

Here are some things to note regarding the resistance distance.

- R_{eff} is non-increasing function of edge weights.
- R_{eff} does not increase when edges are added.
- R^{tot} strictly decreases when edges are added and weights are increased.

Note that these observations are essentially claims about the distance properties of two graphs, call them G and G', when one graph is constructed from the other graph by making changes to one or more edges.

We have said that both geodesic distances and resistances distances are legitimate notions of distances between the nodes on a graph. One might wonder about the relationship between them. In the same way that there are different norms for vectors in $\mathbb{R}^n$, e.g., the ℓ_1, ℓ_2, and ℓ_∞, and those norms have characteristic sizes with respect to each other, so too we can talk about the relative sizes of different distances on nodes of a graph. Here is a theorem relating the resistance distance with the geodesic distance.

Theorem 4. *For R_{eff} and the geodesic distance d:*

1. *$R_{eff}(a,b) = d(a,b)$ iff there exists only one path between a and b.*
2. *$R_{eff}(a,b) < d(a,b)$ otherwise.*

Proof. If there is only one path P between a and b, then $Y_{ij} = Y$, for all ij on this path (by Kirchhoff current law), and $V_i - V_j = YR_{ij}$. It follows that

$$R_{ab} = \frac{V_a - V_b}{Y} = \sum_{(ij)\in P} \frac{V_i - V_i}{Y} = \sum_{(ij)\in P} V_{ij} = d_{ab}.$$

If a path between a and b is added, so that now there are multiple paths between a and b, this new path might use part of the path P. If it does, then call that part of the path P_1; consider the rest of P, and call the shorter of these P_2 and the larger P_3.

Observe that the current through each edge of P_1 is Y; and, in addition, that the current through each edge of P_2 and P_3 is the same for each edge in the path, call them Y_2 and Y_3, respectively. Due to Kirchhoff current law and Kirchhoff circuit/potential law, we have that $Y_2 + Y_3 = Y$ and also that $Y_2, Y_3 > 0$, from which it follows that $Y_2 < Y$. Finally,

$$\begin{aligned} R_{ab} &= \frac{V_a - V_b}{Y} \\ &= \sum_{(ij)\in P_1} \frac{V_i - V_j}{Y} + \sum_{(ij)\in P_2} \frac{V_i - V_j}{Y} \\ &= \sum_{(ij)\in P_1} \frac{V_i - V_j}{Y} + \sum_{(ij)\in P_2} \frac{V_i - V_j}{Y_2} \\ &= \sum_{(ij)\in P_1} R_{ij} + \sum_{(ij)\in P_2} R_{ij} \\ &= d(a,b) \end{aligned}$$

The result follows since R_{eff} doesn't increase when edges are added. □

In a graph that is a tree, there is a unique path between any two vertices, and so we have the following result.

Claim 3. *The metrics R_{eff} and d are the same in a tree. That is, on a tree, $R_{eff}(a,b) = d(a,b)$, for all nodes a and b.*

Fact. R_{eff} can be used to bound several quantities of interest, in particular the commute time, the cover time, etc. We won't go into detail on this.

Here is how R_{eff} behaves in some simple examples.

- Complete graph K_n. Of all graphs, this has the minimum R^{tot}_{eff}: $R^{tot}_{eff}(K_n) = n - 1$.
- Path graph P_n. Among connected graphs, the path graph has the maximum R^{tot}_{eff}: $R^{tot}_{eff}(P_n) = \frac{1}{6}(n-1)n(n+1)$.
- Star S_n. Among trees, this has the minimum R^{tot}_{eff}: $R^{tot}_{eff}(S_n) = (n-1)^2$.

16.3 Extensions to infinite graphs

All of what we have been describing so far is for finite graphs. Many problems of interest have to do with infinite graphs. Perhaps the most basic is whether random walks are recurrent. In addition to being of interest in its own right, considering this question on infinite graphs should provide some intuition for how random walked based spectral methods perform on the finite graphs we have been considering.

Definition 7. *A random walk is* recurrent *if the walker passes through every point with probability* 1*, or equivalently if the walker returns to the starting point with probability* 1*. Otherwise, the random walk is* transient*.*

Note that—if we were to be precise—then we would have to define this for a single node, be precise about which of those two notions we are considering, etc. It turns out that those two notions are equivalent and that a random walk is recurrent for one node iff it is recurrent for any node in the graphs. We'll not go into these details here.

For irreducible, aperiodic random walks on finite graphs, this discussion is of less interest, since a random walk will eventually touch every node with probability proportional to its degree; but consider three of the simplest infinite graphs: $\mathbb{Z}$, $\mathbb{Z}^2$, and $\mathbb{Z}^3$. Informally, as the dimension increases, there are more neighbors for each node and more space to get lost in, and so it should be harder to return to the starting node. Making this precise, i.e., proving whether a random walk on these graphs is recurrent is a standard problem, one version of which appears on HW2.

The basic idea for this that you need to use is to use something called Rayleigh's Monotonicity Law as well as the procedures of shorting and cutting. Rayleigh's Monotonicity Law is a version of the result we described before, which says that R_{eff} between two points a and b varies monotonically with individual resistances. Then, given this, one can use this to do two things to a graph G:

- *Shorting* vertices u and v: this is "electrical vertex identification."
- *Cutting* edges between u and v: this is "electrical edge deletion."

Both of these procedures involve constructing a new graph G' from the original graph G (so that we can analyze G' and make claims about G). Here are the things you need to know about shorting and cutting:

- Shorting a network can only decrease R_{eff}.
- Cutting a network can only increase R_{eff}.

For $\mathbb{Z}^2$, if you short in "Manhattan circles" around the origin, then this only decreases R_{eff}, and you can show that $R_{eff} = \infty$ on the shorted graph, and thus $R_{eff} = \infty$ on the original $\mathbb{Z}^2$. For $\mathbb{Z}^3$, if you cut in a rather complex way, then you can show that $R_{eff} < \infty$ on the cut graph, meaning that $R_{eff} < \infty$ on the original $\mathbb{Z}^3$. This, coupled with the following theorem, establish the result random walks on $\mathbb{Z}^2$ are recurrent, but random walks on $\mathbb{Z}^3$ are transient.

Theorem 5. *A network is recurrent iff* $R_{eff} = \infty$*.*

Using these ideas to prove the recurrence claims is left for HW2: getting the result for $\mathbb{Z}$ is straightforward; getting it for $\mathbb{Z}^2$ is more involved but should be possible; and getting it for $\mathbb{Z}^3$ is fairly tricky—look it up on the web, but it is left as extra credit.