Name: Solutions

Student ID: 31415926

This exam consists of ten pages: this cover page, eight pages each containing one problem, and one blank page. You may use this blank page for scratch paper and/or to write part of the solution to a problem; if you write anything you wish to have graded on this blank page, please indicate that you are doing so. You may use a calculator, and notes on both sides of two standard 8.5-by-11-inch sheets of paper which you have written by hand, yourself. You must show all work other than basic arithmetic.

Each problem is worth 25 points, for a total of 200.

DO NOT WRITE BELOW THIS LINE

Approximate average scores on the eight problems were 18, 18, 18, 21, 18, 16, 18, 21. The average score on the exam was 150, the median 160; for more distributional information see the course webpage. In particular problems 4 and 8 were easier than the others; problem 6 was the hardest.
1. Let $N$ consist of five players, $N = \{1, 2, 3, 4, 5\}$. Define the characteristic function for a game in coalitional form by

$$v(S) = \min\{|S \cap \{1, 2\}|, |S \cap \{3, 4, 5\}|\}.$$ 

This is an instance of the “glove market” game: each of $\{1, 2\}$ owns a left-hand glove and each of $\{3, 4, 5\}$ owns a right-hand glove. If $j$ members of $P$ and $k$ members of $Q$ form a coalition, they get $\min(j, k)$, the number of complete pairs of gloves.

Find the Shapley value of this game.

**Solution:** Use the random-permutation interpretation of the Shapley value. Player 3 (or any of the right-hand players) contributes value to a forming coalition if and only if:

- he enters second, following a left-hand player, or;
- he enters third, following two left-hand players, or;
- he enters fourth, following two left-hand players and a right-hand player.

In these cases he contributes 1. The probability of the first of these is $(2/5) \times (1/4) = 1/10$; the probability of the second is $(2/5) \times (1/4) \times (1/3) = 1/30$. To find the probability of the third, note that this event is the same as entering fourth and being followed by a right-hand player. This has probability $(1/5) \times (2/4) = 1/10$. So $\phi_3(v) = 1/10 + 1/30 + 1/10 = 7/30$. Similarly $\phi_4(v) = \phi_5(v) = 7/30$. Now, the Shapley values for each player must sum to the value of the grand coalition, which is 2, and $\phi_1(v) = \phi_2(v)$. We conclude $\phi_1(v) = \phi_2(v) = (2 - (3 \times 7/30))/2 = 13/20$.

**Discussion:** Very few people came up with the short solution intended here. A lot of people tried more brute-force methods of listing coalitions, with varying degrees of success.
2. Three doctors, Moe, Larry, and Curly, are in partnership and cover for each other. At most one doctor needs to be in the office to see patients at any one time. They advertise their office hours as follows:

Moe: 2pm - 5pm, Curly: 9am - 1pm, Larry: 11am - 4pm.

A coalition is an agreement by one or more doctors as to the times they will really be in the office to see all of their patients. The value of the game, for that coalition, is the number of hours that they collectively save.

(a) [10 points] Find the characteristic function of this game.
(b) [10 points] Find the nucleolus.
(c) [5 points] Assuming that the doctors use the answer from (b) to divide their work, when will they be in the office?

Solution.
(a) The characteristic function is the number of hours saved by a coalition. So \( v(M) = v(C) = v(L) = 0 \) since Moe and Curly’s advertised office hours don’t overlap. \( v(ML) = 2 \) and \( v(CL) = 2 \), since each of these pairs have advertised office hours overlapping by two hours. Finally, \( v(MCL) = 4 \).

(b) The core is the set

\[
\{(x_M, x_C, x_L) : x_M, x_C, x_L \geq 0, x_M + x_C \geq 0, x_M + x_L \geq 2, x_C + x_L \geq 2, x_M + x_C + x_L = 4 \}
\]

We can rewrite this as

\[
\{(x_M, x_C, x_L) : x_M, x_C, x_L \geq 0, x_L \leq 4, x_C \leq 2, x_M \leq 2, x_M + x_C + x_L = 4 \}
\]

Drawing the core in barycentric coordinates as a subset of the triangle with vertices \((0, 0, 4), (0, 4, 0), (4, 0, 0)\), by symmetry we see that the nucleolus is \((x_M, x_C, x_L) = (1, 1, 2)\).

(c) Moe saves one hour and must come in for two hours; Curly saves one hour and must come in for three hours; Larry saves two hours and must come in for three hours. So Curly works from 9am to noon, Larry from noon to 3pm, and Moe from 3pm to 5pm.

Discussion: Many found the characteristic function, although some people worked in terms of “hours spent” in the office instead of “hours saved”. Nobody drew the core, perhaps because I didn’t explicitly ask for it; lots of people just launched straight into calculating the nucleolus. A few computed the Shapley value first as an initial guess for the nucleolus, which is unnecessary.

Also, this problem is funny because Moe, Larry, and Curly are the names of the Three Stooges. They should not be doctors. (If anybody noticed this, I didn’t notice that they noticed it.)
3. Consider the 2-by-3 game of Chomp. The graph of possible positions in this game is given below.

(a) [10 points] Indicate on the graph the Sprague-Grundy value of each position. (Recall that the game ends when one square is left, because that square contains poison.)

(b) [15 points] I play Chomp with multiple chocolate bars. On each turn I take a rectangular “chomp” out of one chocolate bar, removing a square and all squares above or to the right of it. I start with the position (consisting of five partially eaten chocolate bars)

Give all winning moves, if any, from this position.

Solution.

(a) Let \((k, l)\) denote the position with \(k\) squares in the first row and \(l\) in the second row; \(k \geq l\). Then

\[ g(1, 0) = g(2, 1) = g(3, 2) = 0, g(2, 0) = g(1, 1) = 1, g(3, 0) = g(2, 2) = 2, g(3, 1) = 3, g(3, 3) = 4 \]

by repeated application of the definition of Sprague-Grundy function.

(b) The five bars in the position have value 0, 0, 2, 2, 3. This position therefore has value 0 \(\oplus\) 0 \(\oplus\) 2 \(\oplus\) 2 \(\oplus\) 3 = 3. Winning moves are therefore those that take any summand \(x\) to \(x \oplus 3\).

These are taking \((3, 2)\) to \((3, 1)\); \((3, 0)\) to \((2, 0)\); \((2, 2)\) to \((2, 0)\) or \((1, 1)\); \((3, 1)\) to \((2, 1)\). In particular there are now inning moves that involve acting on \((1, 0)\), which is hardly surprising as the only possible move there is to eat the poison. (Formally, this move isn’t even allowed!)

Discussion. For (a), people basically knew how to compute the Sprague-Grundy function, although there were the inevitable mistakes in calculation. (b) was fairly straightforward except that some people didn’t realize this was a sum of games and tried to compute a winning move from each chocolate bar individually.

Now, let the payoffs depend on the options chosen. In particular:
- paper pays $x$ to scissors;
- scissors pays $y$ to rock;
- rock pays $z$ to paper.

$x$, $y$, and $z$ are all positive.

Write this as a matrix game, and solve it.

**Solution.** The game has matrix

\[
\begin{pmatrix}
0 & -z & y \\
z & 0 & -x \\
-y & x & 0
\end{pmatrix}
\]

where the strategies are listed in the order rock, paper, scissors.

The game is symmetric, so it has value 0. Both players have the same optimal strategy, which we denote $(p_R, p_P, p_S)$. Since the expected payoff for the first player picking rock must be zero, we have $-zp_P + yp_S = 0$; thus $p_P = (y/z)p_S$. Similarly, looking at the second row, $p_R = (x/z)p_S$. But $p_R + p_P + p_S = 1$, so we have

\[
\frac{x}{z}p_S + \frac{y}{z}p_S + p_S = 1.
\]

We can rewrite this as $(x + y + z)/zp_S = 1$. So $p_S = z/(x + y + z)$. Back-substituting, $p_R = x/(x + y + z)$ and $p_P = y/(x + y + z)$.

**Discussion.** The easiest problem, overall, on the exam. The big difficulty on this one was that some people decided that since the game was symmetric, it must have solution $(1/3, 1/3, 1/3)$. This isn’t true; that’s not the kind of symmetry we have here. (Did you honestly believe that? Or were you just trying for partial credit?)
5. Solve the stochastic game with two positions

\[ G^{(1)} = \begin{pmatrix} 3 & 1 + \frac{1}{2} G^{(2)} \\ 0 & \frac{1}{2} G^{(2)} \end{pmatrix}, \quad G^{(2)} = \begin{pmatrix} 2 & 0 \\ \frac{1}{2} G^{(1)} - 1 & \frac{1}{2} G^{(1)} + 2 \end{pmatrix}. \]

That is, give the value of the games \( G^{(1)} \) and \( G^{(2)} \), and the optimal strategies for each player in \( G^{(1)} \) and in \( G^{(2)} \).

**Solution.** We need to solve the system of equations

\[ v_1 = Val \left( \begin{pmatrix} 3 & 1 + v_2/2 \\ 0 & v_2/2 \end{pmatrix} \right), \quad v_2 = Val \left( \begin{pmatrix} 2 & 0 \\ v_1/2 - 1 & v_1/2 + 2 \end{pmatrix} \right). \]

We guess that the first matrix here has a saddle point at the upper right (this is true unless \( v_2 > 4 \), which seems unlikely) and that the second matrix has no saddle point (true if \( -4 < v_1 < 6 \)). We will need to check these assumptions when we finish.

This gives the system

\[ v_1 = 1 + \frac{1}{2} v_2, \quad v_2 = \frac{v_1 + 4}{5}. \]

Substituting the first equation into the second gives

\[ v_2 = \frac{(1 + v_2/2) + 4}{5} \]

which can be solved to give \( v_2 = 10/9 \) and thus \( v_1 = 14/9 \). Our original assumptions regarding saddle points are seen to be correct.

Finally, the optimal strategies in game 1 are \( \vec{p}_1 = (1, 0), \vec{q}_1 = (0, 1) \), since the upper right corner is a saddle point. The optimal strategies in game 2 are \( \vec{p}_2 = (3/5, 2/5), \vec{q}_2 = (5/9, 4/9) \) from the usual formulas for 2-by-2 games, applied to the matrix \( \begin{pmatrix} 2 & 0 \\ -2/9 & 25/9 \end{pmatrix} \).

**Discussion.** The most common error here – and this was disappointingly common – was to assume that both matrices had saddle points and proceed blindly.
6. Find the NTU-solution, threat point, and equilibrium exchange rate of the bimatrix game

\[
\begin{pmatrix}
(3, -2) & (1, -1) & (4, -3) \\
(6, 1) & (0, 8) & (3, 7)
\end{pmatrix}.
\]

**Solution.** Consider this as the bimatrix game \((A, B)\), with

\[
A = \begin{pmatrix} 3 & 1 & 4 \\ 6 & 0 & 3 \end{pmatrix}, \quad -B = \begin{pmatrix} 2 & 1 & 3 \\ -1 & -8 & -7 \end{pmatrix}.
\]

Then \(A\) and \(-B\) both have saddle points in the first row, second column; the value there, \((1, -1)\), is the threat point.

We now want to maximize \((x - 1)(y + 1)\) along the Pareto frontier. The Pareto frontier consists of the two line segments from \((1, \frac{7}{2})\) to \((3, 7)\) and from \((3, 7)\) to \((6, 1)\). Let \(f(x)\) be the value of \((x - 1)(y + 1)\) at the point on the Pareto frontier with a given \(x\)-coordinate, where \(x\) ranges from 1 to 6.

The first of these segments has equation \(y = 8 - x/3\), so \(f(x) = (x - 1)(9 - x/3) = \frac{(x - 1)(27 - x)}{3} = \frac{-x^2 + 28x - 27}{3}\) for \(1 \leq x \leq 3\).

The second segment has equation \(y = 13 - 2x\), so \(f(x) = (x - 1)(14 - 2x) = \frac{-2x^2 + 16x - 14}{3}\) for \(3 \leq x \leq 6\).

We can therefore check that \(f'(x) = \frac{-2x + 28}{3}\) for \(1 \leq x \leq 3\) and \(f'(x) = \frac{-4x + 16}{3}\) for \(3 \leq x \leq 6\). So \(f\) is increasing for \(1 \leq x \leq 4\) and decreasing for \(4 \leq x \leq 6\) and has a maximum at \(x = 4\). This is the \(x\)-coordinate of the NTU-solution; the \(y\)-coordinate is \(13 - 2(4) = 5\). The equilibrium exchange rate is the slope of the line from the threat point \((1, -1)\) to the NTU solution \((4, 5)\), which is 2. (Alternatively, it’s the negative reciprocal of the slope of the Pareto frontier at the NTU solution.)

**Discussion.** Common errors here were to maximize \(xy\) instead of \((x - 1)(y + 1)\), or to only maximize along one of the two line segments which make up the Pareto frontier. Also, the equilibrium exchange rate is the slope of the line from the threat point to the NTU solution, not the slope of the line from the origin to the NTU solution.
7. Two bears each chase one of two ducks. The values of the ducks to the bears are 2 and \(x\), where \(x > 2\). If the bears catch the same duck, they share it, with the first bear receiving a proportion \(r\) of the total value, where \(0 < r < 1\). 

(a) [15 points] Write the bimatrix of this game and find all the pure strategic equilibria. Your answer should depend on \(r\) and \(x\).

(b) [10 points] Let \(x = 3\) and \(r = 3/5\). Find the equalizing strategic equilibrium.

**Solution.**

(a) The game has bimatrix

\[
\begin{pmatrix}
(2r, 2s) & (2, x) \\
(x, 2) & (xr, xs)
\end{pmatrix}
\]

where \(s = 1 - r\). A PSE is a column maximum for player I that’s also a row maximum for player II. Therefore:

- the upper left is never a PSE, since \(2s < 2\) always;
- the lower right is never a PSE, since \(xs < x\) always;
- the upper right is always a row maximum for II (\(x > 2s\)), so is a PSE when it’s a column maximum for I, that is, when \(xr < 2\).
- the lower left is always a column maximum for I (\(x > 2 > 2r\)), so is a PSE when it’s a row maximum for II, that is, when \(xs < 2\).

(The bears consistently chase after different ducks if sharing the larger duck wouldn’t help them.)

(b) The resulting game has bimatrix

\[
\begin{pmatrix}
(1.2, .8) & (2, 3) \\
(3, 2) & (1.8, 1.2)
\end{pmatrix}
\]

To find the ESE, player I should play in player II’s matrix \(\begin{pmatrix} .8 & 3 \\ 2 & 1.2 \end{pmatrix}\) – this gives the equalizing strategy \((4/15, 11/15)\). Player II should play in player I’s matrix \(\begin{pmatrix} 1.2 & 2 \\ 3 & 1.8 \end{pmatrix}\), giving the equalizing strategy \((1/10, 9/10)\).

**Discussion.** There were no really common errors for this problem; people that lost points despite making a serious attempt at it usually got tripped up somewhere in the case analysis in part (a).
8. Find the equivalent strategic form and solve:
(There was a tree here, drawn by hand. Sorry, I don’t have a scanner.)

**Solution.** Possible strategies for player I are AC, AD, BC, BD; for player II, a and b.

Playing out these strategies in the tree, we get the strategic form:

\[
\begin{pmatrix}
\frac{1}{3} \cdot 6 + \frac{2}{3} \cdot 3 & \frac{1}{3} \cdot 9 + \frac{2}{3} \cdot 3 \\
\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 3 & \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 3 \\
\frac{1}{3} \cdot 7 + \frac{2}{3} \cdot 4 & \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 4 \\
\frac{2}{3} \cdot 7 + \frac{1}{3} \cdot 4 & \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 4
\end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 2 & 3 \\ 5 & 3 \end{pmatrix}.
\]

The second row is dominated by the third row; the fourth row is also (weakly) dominated by the third row. So we have to solve the 2-by-2 game

\[
\begin{pmatrix} 4 & 5 \\ 5 & 3 \end{pmatrix}.
\]

The solution is \( \vec{p} = \vec{q} = (2/3, 1/3), V = 13/3 \). This translates back to player I picking strategy AC with probability 2/3 and BC with probability 1/3; player II picks strategy a with probability 2/3 and b with probability 1/3. (In particular, the D-branches are never followed, which isn’t surprising as both of them have lower value than the corresponding C-branch.)

**Discussion.** A lot of people noticed that BC and BD always have identical results and therefore didn’t bother to separate the two strategies; this gives a three-by-two matrix.