Lecture 14: Likelihoods, Likelihood Ratios, Hypotheses Testing and the Neyman-Pearson Lemma

Example: You have two coins, one gold, one bronze:

The gold coin comes up heads 90% of the time
The bronze coin comes up heads 10% of the time

One of these two coins is chosen by the toss of a fair coin. The coin chosen is tossed and a head comes up. What is the probability that the gold coin was chosen?

\[ \begin{array}{c|c|c|c|c}
 & \text{Gold} & \text{Bronze} \\
\hline
\text{H} & 0.9 & 0.1 \\
\text{T} & 0.1 & 0.9 \\
\end{array} \]

\[ G = \text{gold coin chosen} \]
\[ B = \text{bronze coin chosen} \]

Solution:

\[ \Pr(G|H) = \frac{\Pr(G \& H)}{\Pr(H)} = \frac{\Pr(G) \Pr(H|G)}{\Pr(H)} \]

Now,

\[ \Pr(H) = \Pr(G \& H) + \Pr(B \& H) \quad \text{why?} \]
\[ = \Pr(G) \Pr(H|G) + \Pr(B) \Pr(H|B) \]
\[ = 0.5 \times 0.9 + 0.5 \times 0.1 = 0.5 \]

Therefore,

\[ \Pr(G|H) = \frac{0.5 \times 0.9}{0.5} = 0.9 = \Pr(H|G) \]
\[ \Pr(B|H) = \frac{0.5 \times 0.1}{0.5} = 0.1 = \Pr(H|B) \]

Now suppose \( \Pr(G) = 0.9 \) and \( \Pr(B) = 0.1 \). Then,

\[ \Pr(G|H) = \frac{0.9 \times 0.9}{0.9 \times 0.9 + 0.1 \times 0.1} = 0.99 \]
\[ \Pr(G|H) = \frac{0.1 \times 0.1}{0.9 \times 0.9 + 0.1 \times 0.1} = 0.01 \]
Again, suppose \( \Pr(G) = 0.1 \) and \( \Pr(B) = 0.9 \). Then,

\[
\Pr(G|H) = \frac{0.9 \times 0.1}{0.9 \times 0.1 + 0.1 \times 0.9} = \frac{1}{2}
\]

\[
\Pr(B|H) = \frac{1}{2} \quad \text{similarly}
\]

Conclusion:

\( \Pr(G|H) \) depends critically on \( \Pr(G) \). If \( \Pr(G) = \Pr(B) = \frac{1}{2} \) (equally probable a priori) then:

\[
\begin{align*}
\Pr(G|H) &= \Pr(H|G) \\
\Pr(B|H) &= \Pr(H|B),
\end{align*}
\]

relations that are usually false.

**Definition**

The function \( Pr(H|\cdot) \), defined by \( G \mapsto \Pr(H|G) \) and \( B \mapsto \Pr(H|B) \), is called the **likelihood function** generated by the observation of a head. Likelihood is probability of a fixed event (here \( H \)) viewed as a function of hypotheses (here \( G, B \)).

Suppose we have a coin which has either probability \( p_0 = 0.1 \) or \( p_1 = 0.9 \) of coming up heads. It is tossed 10 times and we obtain 4 heads and 6 tails. Is \( p = p_0 \) or \( p = p_1 \)?

With prior probabilities of the hypotheses \( p_0, p_1 \) we can proceed as in the previous example:

\[
\Pr(p_0|4\text{Hs}, 6\text{Ts}) = \frac{\Pr(4\text{Hs}, 6\text{Ts}|p_0) \Pr(p_0)}{\cdots + \cdots}
\]

Without prior probabilities, the conventional solution is to choose one of \( p_0, p_1 \) to be the null hypothesis, and test that. For example, if \( H_0 \) is \( p = p_0 \), we test against the alternative \( H_1: p = p_1 \). Intuitively, we should reject the null if we have "too many" heads. Is 4 too many? Fix the critical region \( C = \{ \# \text{ heads} \geq c \} \), and choose \( c \) so that \( \Pr(C|p_0) = 0.05 \) (say). If we do this, \( c = 3 \) corresponds to 0.07 , \( c = 4 \) to 0.015.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \Pr(k \text{ heads}) )</th>
<th>( \Pr(\leq k \text{ heads}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.348</td>
<td>0.348</td>
</tr>
<tr>
<td>1</td>
<td>0.387</td>
<td>0.735</td>
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<tr>
<td>2</td>
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<td>0.923</td>
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<td>3</td>
<td>0.057</td>
<td>0.985</td>
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<tr>
<td>4</td>
<td>0.011</td>
<td>0.996</td>
</tr>
<tr>
<td>5</td>
<td>0.001</td>
<td>0.997</td>
</tr>
</tbody>
</table>
The loaded die

A die is either fair \(p_0\) or biased in the manner indicated by \(p_1\)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_0):</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{6})</td>
</tr>
<tr>
<td>(p_1):</td>
<td>(\frac{1}{5})</td>
<td>(\frac{7}{40})</td>
<td>(\frac{7}{40})</td>
<td>(\frac{7}{40})</td>
<td>(\frac{7}{40})</td>
<td>(\frac{1}{10})</td>
</tr>
</tbody>
</table>

It is rolled 10 times and the number \(n_1\) of aces and \(n_6\) of 6s is noted. The joint distribution of \(n_1\) and \(n_6\) based on \(p_1\) is

\[
\Pr(n_1 = x, n_6 = y) = \frac{10!}{x!y!(10 - x - y)!} \left( \frac{1}{5} \right)^x \left( \frac{7}{10} \right)^{10-x-y} \left( \frac{1}{10} \right)^y
\]

Suppose we want to test \(H_0: p = p_0\) against \(H_1: p = p_1\). What test statistic should be used?

Options:
(a) Reject \(H_0\) if “too many 1s”, i.e. large \(n_1\)
(b) Reject \(H_0\) if “too few 6s”, i.e. small \(n_6\)
(c) Some other rejection rule, perhaps based on \(n_1\) and \(n_6\) jointly.

Which, if any, is “best” and how do we decide in general?

Look at (a): If we reject \(H_0\) when \(n_1 \geq c\) we have Type 1 error of:

\[
\sum_{x \geq c} \binom{10}{x} \left( \frac{1}{6} \right)^x \left( \frac{5}{6} \right)^{10-x}
\]

<table>
<thead>
<tr>
<th>(k)</th>
<th>(\Pr(x_1 = k))</th>
<th>(\Pr(x_1 \leq k))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1615</td>
<td>0.1615</td>
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<tr>
<td>1</td>
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<td>6</td>
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<td>0.9997</td>
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<tr>
<td>9</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>10</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Look at (c): the likelihood ratio

\[
\frac{\frac{10!}{x!y!(10-x-y)!} \left( \frac{1}{5} \right)^x \left( \frac{7}{10} \right)^{10-x-y} \left( \frac{1}{10} \right)^y}{\frac{10!}{x!y!(10-x-y)!} \left( \frac{1}{6} \right)^x \left( \frac{2}{3} \right)^{10-x-y} \left( \frac{1}{6} \right)^y} = \frac{L_1}{L_0}, \text{ say}
\]
\[
\log \frac{L_1}{L_0} = x \log \left( \frac{1}{6} \times \frac{2}{10} \right) + y \log \left( \frac{1}{12} \times \frac{3}{10} \right) \\
= x \log \frac{120}{105} + y \log \frac{12}{21} \\
10 \log \frac{L_1}{L_0} = 1.3x - 5.6y = l
\]

We could calculate a cutoff for \( l \), e.g. \( l \geq c \), where \( c \) is such that \( \Pr(l \geq c | \theta_0) \approx 0.05 \). Then \( \Pr(l \geq c | \theta_1) \) is the power (to reject, correctly).

This development is continued in Homework #4.

**Neyman-Pearson Lemma**

Suppose that the likelihood ratio test that rejects \( H_0 \) when \( l \geq c \), has significance level \( \alpha \). Then any other test which has significance level \( \alpha^* \leq \alpha \) has power less than or equal to that of the likelihood ratio test.