Problem 9.6 (p.437)

(c) 1, all the linear terms and all the quadratic terms are linearly independent and form a basis of the space of quadratic polynomials. Thus,

\[
\text{the dimension} = 1 + \#(x_i) + \#(x_i^2) + \#(x_ix_j)
\]
\[
= 1 + M + M(M - 1)/2
\]
\[
= 1 + 3M/2 + M^2/2
\]

(a) \( M = 2 \), dim = 6.
(b) \( M = 4 \), dim = 15.

Problem 9.8 (p.437)

(a) \[
(b_{10} + b_{11}x_1)(b_{20} + b_{21}x_2)(b_{30} + b_{31}x_3)
\]
\[
= b_{10}b_{20}b_{30} + b_{11}b_{20}b_{30}x_1 + b_{10}b_{21}b_{30}x_2 + b_{10}b_{20}b_{31}x_3 + b_{11}b_{21}b_{30}x_1x_2
\]
\[
+ b_{11}b_{20}b_{31}x_1x_3 + b_{10}b_{21}b_{31}x_2x_3 + b_{11}b_{21}b_{31}x_1x_2x_3
\]

That is, every function in \( C \) is a linear combination of 1, \( x_1 \), \( x_2 \), \( x_3 \), \( x_1x_2 \), \( x_1x_3 \), \( x_1x_2x_3 \), and also it’s easy to check that all these functions belong to \( C \). By Thm. 9.1, these functions are linearly independent. Thus, they form a basis of the span of \( C \).

(b) Suppose \( C \) is a linear space. Now \( 1 + x_1x_2x_3 \) must belong to \( C \).
\[
1 + x_1x_2x_3 = (b_{10} + b_{11}x_1)(b_{20} + b_{21}x_2)(b_{30} + b_{31}x_3)
\]
\[
\begin{align*}
b_{10}b_{20}b_{30} &= 1 \quad (1) \\
b_{11}b_{21}b_{31} &= 1 \quad (2) \\
b_{11}b_{20}b_{30} &= 0 \quad (3) \\
& \vdots
\end{align*}
\]

It follows the first two equations that \( b_{20} \neq 0 \), \( b_{30} \neq 0 \) and \( b_{11} \neq 0 \), and hence that \( b_{11}b_{20}b_{30} \neq 0 \), which contradicts (3). Therefore, \( C \) is not linear space.

Problem 9.13 (p.438)

(a) \( I_1 + I_2 + I_3 + I_4 = 1 \Rightarrow I_4 = 1 - I_1 - I_2 - I_3 \).
Each of \( I_1, I_2, I_3, I_4 \) is a linear combination of 1, \( I_1, I_2, I_3 \). Therefore, 1, \( I_1, I_2, I_3 \)
form a basis of $G$.

(b) \[(I_1 I_2 I_3)^T = A (I_1 I_2 I_3 I_4)^T\]

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

(c) \[(I_1 I_2 I_3 I_4)^T = B (I_1 I_2 I_3)^T\]

\[
B = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & -1 & -1 & -1
\end{pmatrix}
\]

It’s easy to check $AB = I$. Thus, $A = A^{-1}$.

**Problem 9.17 (p.446)**

(a) Suppose $G$ is nonidentifiable, there is a nonzero cubic polynomial $b_0 + b_1x + b_2x^2 + b_3x^3$ which equals 0 on the design set. If $b_3 = 0$, $d \leq 2$ by example 9.10. Otherwise, the cubic polynomial has at most three distinct roots, $d \leq 3$.

Suppose, conversely, $d \leq 3$. For $d = 1$, $d = 2$ and $d = 3$ respectively, $x - x'_1$, $(x - x'_1)(x - x'_2)$ and $(x - x'_1)(x - x'_2)(x - x'_3)$ are nonzero functions that equal to 0 on the design set. Thus, $G$ is nonidentifiable.

(b) $g(x) = -(x - 1)(x - 2)(x - 3)/6$.

(c) 

![Figure 1: Plot of the function](image-url)
Problem 9.22 (p.447)

(a) Let \( g(x) = 2x_1 + 3x_2 - x_3 \), then \( g(x) \) is a nonzero function that equals to zero on design set. Thus, \( G \) is nonidentifiable.

(b) Let \( g(x) = b_0 + b_1x_1 + b_2x_2 \).

\[
\begin{align*}
  b_0 - 2b_1 &= 0 \quad (1) \\
  b_0 + b_2 &= 0 \quad (2) \\
  b_0 + b_1 + b_2 &= 0 \quad (3) \\
  b_0 + b_1 - 2b_2 &= 0 \quad (4)
\end{align*}
\]

Solve the system of equations, \( b_0 = 0, b_1 = 0, b_2 = 0 \) is the only solution. Thus, \( G \) is identifiable.

Problem 9.25 (p.447)

(a) \( X = \begin{pmatrix} 1 & 0 & 0 \\
                          0 & 1 & 0 \\
                          0 & 0 & 1 \end{pmatrix} \)  
(b) \( \tilde{X} = \begin{pmatrix} 1 & -1 & 1 \\
                                1 & 0 & 0 \\
                                1 & 1 & 1 \end{pmatrix} \)  
(c) \( A = \begin{pmatrix} 1 & 1 & 1 \\
                        -1 & 0 & 1 \\
                         1 & 0 & 1 \end{pmatrix} \) It’s easy to check \( \tilde{X} = XA^T \).

Problem 9.27 (p.453)

Given \( G \) is saturated, suppose there exists one set \( X_{i_0} \) has more than one points, say \( x'_s \) and \( x'_t \). Since \( I_{x'_s}(x'_s) = 1 \) and \( I_{x'_s}(x'_t) = 0 \), then \( I_{x'_s} \) in not in \( G \). This contradicts with \( G \) is saturated. Thus, in each set, there is at most one element.

Conversely, suppose each of the sets in the partition contains at most one design point, then \( I_j \in G \) and satisfy the conditions that \( I_j \) takes 1 on the design point \( i \) and takes 0 on all other design points. By Thm. 9.11, \( G \) is saturated.

Problem 9.31 (p.454)

(a) \( X = \begin{pmatrix} 1 & -3 & 9 & -27 \\
                          1 & -1 & 1 & -1 \\
                          1 & 1 & 1 & 1 \\
                          1 & 3 & 9 & 27 \end{pmatrix} \)

(b) \( c = (1 2 2 0)^T \). \( b = X^{-1}c = (105 1 - 9 - 1)^T/48 \).

Thus, \( g(x) = (105 + x - 9x^2 - x^3)/48 \).
Problem 9.36 (p.454)

(a) \[ X = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix} \]

(b) \[ X^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
1 & -1 & -1 & 1
\end{pmatrix} \]

That is \( X \) is invertible, by (Thm. 9.15), \( G \) is identifiable and saturated.

Problem 9.42 (p.468)

(a) 1, \( x \), \( x^2 \), \( x^3 \) is a basis and clearly can be written as linear combinations of 1, \( x \), \( x^2 - c_1 \), \( x^3 - c_2x \), thus 1, \( x \), \( x^2 - c_1 \), \( x^3 - c_2x \) is also a basis.

\[
\begin{align*}
<1, x> &= r \cdot 1 \cdot -3 + r \cdot 1 \cdot -1 + r \cdot 1 \cdot 1 + r \cdot 1 \cdot 3 = 0 \\
<1, x^2 - c_1> &= r(1 \cdot (9 - c_1) + 1 \cdot (1 - c_1) + 1 \cdot (1 - c_1) + 1 \cdot (9 - c_1)) = 0 \\
\Rightarrow c_1 &= 5 \\
<1, x^3 - c_2x> &= r(-3 \cdot (-27 + 3c_2) - 1 \cdot (-1 + c_2) + 1 \cdot (1 - c_2) + 3 \cdot (27 - 3c_2)) = 0 \\
\Rightarrow c_2 &= 41/5
\end{align*}
\]

1, \( x \), \( x^2 - 5 \), \( x^3 - 8.2x \) is the orthogonal basis.

(b) 
\[
\begin{align*}
\|1\|^2 &= r + r + r + r = 4r \\
\|x\|^2 &= r \cdot 9 + r \cdot 1 + r \cdot 1 + r \cdot 9 = 20r \\
\|x^2 - 5\|^2 &= r \cdot 16 + r \cdot 16 + r \cdot 16 + r \cdot 16 = 64r \\
\|x^3 - 8.2x\|^2 &= r((-27 + 24.6)^2 + (-1 + 8.2)^2 + (1 - 8.2)^2 + (27 - 24.6)^2) = 115.2r
\end{align*}
\]

(c)

Figure 2: Plot of the functions of the orthogonal basis
Problem 9.45 (p.468)

Name the columns in Table 9.5 \(c_1, \cdots, c_7\) respectively. And denote the column of 1’s as \(c_0\). And it’s easy to check \(<c_i, c_j> = 0\) for \(\forall i, j \in 0, \cdots, 7\) and \(<c_i, c_i> = 8r\).

(a) Design matrix \(X = [c_0 \cdots c_7]\).

(b) \(<x_i, x_j> = <c_i, c_j>\), so the gram matrix \(M = 8r \cdot I\).

(c) All off-diagonal elements in gram matrix are zero, so the given basis is orthogonal.

(d) The squared norms of these functions are the diagonal elements in the gram matrix, they are all \(8r\).

(e) Since \(G\) has orthogonal basis consisting of functions having positive norm, it’s identifiable. Furthermore, since the dimension of \(G\) coincides with the number of design points (Thm 9.10), this space is saturated.

Problem 9.53 (p.470)

\[(a) \quad X = \begin{pmatrix} 1 & -2 & 1 & -3 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}, \quad M = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 10 & 0 & 14 \\ 0 & 0 & 4 & 0 \\ 0 & 14 & 0 & 20 \end{pmatrix}.\]

Therefore, the basis is not orthogonal. \(M\) is invertible, so \(G\) is identifiable. Furthermore, since the dimension of \(G\) coincides with the number of design points (Thm 9.10), this space is saturated.

\[(b) \quad X = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 4 \end{pmatrix}.\]

Therefore, the basis is orthogonal. \(M\) is invertible, so \(G\) is identifiable. Since \(\dim(G) = 3 < d\), \(G\) is not saturated.

Problem 9.55 (p.483)

\[(a) \quad \|P_G h\|^2 = <P_G h, P_G h> = <P_G h, h> + <P_G h, h - P_G h> = <P_G h, h> \]
Thus, \( \| h - P_G h \|^2 = \| h \|^2 - \| P_G h \|^2 = \| h \|^2 - \langle P_G h, h \rangle \).

(b)
\[
\langle h_1, P_G h_2 \rangle = \langle P_G h_1, h_2 \rangle + \langle h_1, P_G h_2 \rangle
\]
\[
= \langle P_G h_1, P_G h_2 \rangle + \langle h_1, h_2 \rangle - \langle P_G h_1, h_2 - P_G h_2 \rangle
\]
\[
= \langle P_G h_1, h_2 \rangle - \langle h_1, h_2 - P_G h_2 \rangle
\]
\[
= \langle P_G h_1, h_2 \rangle
\]

**Problem 9.60 (p.484)**

(a) Since \( g_i \in G_0, \langle g_i, g - P_{G_0} g \rangle = 0 \) \((i = 1, \cdots, p - 1)\).

Because \( g \) is in \( G \) but not in \( G_0 \) and \( G \) is identifiable, \( \| g - P_{G_0} g \| > 0 \).

That is \( g_1, \cdots, g_{p-1}, g - P_{G_0} g \) is a set of orthogonal functions with positive norms.

Considering \( \dim(G) = p \), they form the orthogonal basis of \( G \).

(b) By (a), \( g_1, \cdots, g_{p-1}, g - P_{G_0} g \) is an orthogonal basis of \( G \). And it’s given that \( \| g_i \| = 1, (i = 1, \ldots, p - 1) \). As mentioned before, \( \| g - P_{G_0} g \| > 0 \), then
\[
\frac{\| g - P_{G_0} g \|}{\| g - P_{G_0} g \|} = 1
\]

Therefore,
\[
g_1, \cdots, g_{p-1}, \frac{g - P_{G_0} g}{\| g - P_{G_0} g \|}
\]
is an orthogonal basis.

**Problem 9.67 (p.484)**

(a) and (b) are Thm 9.25 applied to 1-dimensional space, and (c) is the Pythagorean theorem applied to this special case.

**Problem 9.71 (p.491)**

(a) \( h(-3) = h(3) = -\sqrt{2}/2, h(-1) = h(1) = \sqrt{2}/2 \).

(b)
\[
\langle 1, h \rangle = h(-3) + h(-1) + h(1) + h(3) = 0
\]
\[
\langle x, h \rangle = -3h(-3) - h(-1) + h(1) + 3h(3) = 0
\]
\[
\langle x^2, h \rangle = 9h(-3) + h(-1) + h(1) + 9h(3) = -8\sqrt{2}
\]
(c) $1, x, x^2$ form the basis.

$$X = \begin{pmatrix} 1 & -3 & 9 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{pmatrix} \quad M = X^TX = \begin{pmatrix} 4 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 164 \end{pmatrix} \quad U = \begin{pmatrix} 0 \\ 0 \\ -8\sqrt{2} \end{pmatrix}$$

The normal equations are:

$$4b_0^* + 20b_2^* = 0$$
$$20b_1^* = 0$$
$$20b_0^* + 164b_2^* = -8\sqrt{2}$$

(d) Solve the equations: $b_0^* = 5\sqrt{2}/8$, $b_1^* = 0$, $b_2^* = -\sqrt{2}/8$.

(e) $\|h\|^2 = h(-3)^2 + h(-1)^2 + h(1)^2 + h(3)^2 = 2$.

(f) $\|h^*\|^2 = b^TU = 2$. $\|h - h^*\|^2 = \|h\|^2 - \|h^*\|^2 = 2 - 2 = 0$.

**Problem 9.74 (p.492)**

The normal equations are:

$$< 1, 2 > b_0^* + < 1, g_1 - \bar{g}_1 > b_1^* + < 1, g_2 - \bar{g}_2 > b_2^* = < 1, h > \quad (1)$$
$$< g_1 - \bar{g}_1, 2 > b_0^* + < g_1 - \bar{g}_1, g_1 - \bar{g}_1 > b_1^* + < g_1 - \bar{g}_1, g_2 - \bar{g}_2 > b_2^* = < g_1 - \bar{g}_1, h > \quad (2)$$
$$< g_2 - \bar{g}_2, 1 > b_0^* + < g_2 - \bar{g}_2, g_1 - \bar{g}_1 > b_1^* + < g_2 - \bar{g}_2, g_2 - \bar{g}_2 > b_2^* = < g_2 - \bar{g}_2, h > \quad (3)$$

(a) Notice

$$< 1, 1 > = w$$
$$< 1, g_1 - \bar{g}_1 > = w \bar{g}_1 - \bar{g}_1 < 1, 1 > = 0$$
$$< 1, g_2 - \bar{g}_2 > = w \bar{g}_2 - \bar{g}_2 < 1, 1 > = 0$$

Then equation (1) degenerates into $b_0^* = < 1, h > / < 1, 1 > = \bar{h}$.

(b) $< 1, g_1 - \bar{g}_1 > = < 1, g_2 - \bar{g}_2 > = 0$. Then equation (2) and (3) degenerate into:

$$< g_1 - \bar{g}_1, g_1 - \bar{g}_1 > b_1^* + < g_1 - \bar{g}_1, g_2 - \bar{g}_2 > b_2^* = < g_1 - \bar{g}_1, h >$$
$$< g_2 - \bar{g}_2, g_1 - \bar{g}_1 > b_1^* + < g_2 - \bar{g}_2, g_2 - \bar{g}_2 > b_2^* = < g_2 - \bar{g}_2, h >$$

**Problem 9.81 (p.493)**

(a) $M = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 10 & 0 & 14 \\ 0 & 0 & 4 & 0 \\ 0 & 14 & 0 & 20 \end{pmatrix}, \quad U = \begin{pmatrix} < 1, x_3^2 > \\ < x_1, x_3^2 > \\ < x_2, x_3^2 > \\ < x_3, x_3^2 > \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \\ 16 \\ 0 \end{pmatrix}$

Thus, the normal equations are:

$4b_0^* = 20$, $10b_1^* + 14b_3^* = 0$, $4b_2^* = 16$, $14b_1^* + 20b_3^* = 0$.

Solve the equations: $b_0^* = 5$, $1b_1^* = 0$, $b_2^* = 4$, $b_3^* = 0$. 
Therefore, the least square approximation $h^* = 5 + 4x_2$.

(b) Based on the gram matrix shown before, $1, x_1, x_2$ are pairwise orthogonal, and $1, x_2$ are orthogonal to $x_3$.

$$x_3 - P_{[1,x_1,x_2]}x_3 = x_3 - P_{[x_1]}x_3 = x_3 - \frac{<x_1,x_3>}{\|x_1\|^2}x_1 = x_3 - \frac{7}{5}x_1$$

Thus, the normal equations are:

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} <1,x_3^2> \\ <x_1,x_3^2> \\ <x_2,x_3^2> \\ <x_3-1.4x_1,x_3^2> \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \\ 16 \\ 0 \end{pmatrix}$$

Thus, the normal equations are: $4b_0^* = 20, 10b_1^* = 0, 4b_2^* = 16, 4b_3^* = 0$. Solve the equations: $b_0^* = 5, b_1^* = 0, b_2^* = 4, b_3^* = 0$.

Therefore, the least square approximation $h^* = 5 + 4x_2$. 