3. Heights and weights of a large group of people follow a bivariate normal distribution, with correlation 0.75. Of the people in the 90th percentile of weights, about what percentage are above the 90th percentile of heights?

4. Suppose X and Y are standard normal variables. Find an expression for $P(X+2Y \leq 3)$ in terms of the standard normal distribution function Φ,

a) in case X and Y are independent;

b) in case X and Y have bivariate normal distribution with correlation 1/2.

5. Let X and Y have bivariate normal distribution with parameters μ_X, μ_Y, ${\sigma_X}^2$, ${\sigma_Y}^2$, and ρ. Let $P(X > \mu_X, Y > \mu_Y) = q$. Find:

a) a formula for q in terms of ρ; b) a formula for ρ in terms of q.

6. Let X and Y be independent standard normal variables.

a) For a constant k, find $P(X > kY)$.

b) If $U = \sqrt{3}X + Y$, and $V = X - \sqrt{3}Y$, find $P(U > kV)$.

c) Find $P(U^2 + V^2 < 1)$.

d) Find the conditional distribution of X given $V = v$.

7. Let X and Y have bivariate normal distribution with parameters μ_X, μ_Y, ${\sigma_X}^2$, ${\sigma_Y}^2$, and ρ.

a) Show that X and Y are independent if and only if they are uncorrelated.

b) Find $E(Y|X = x)$. c) Find $Var(Y|X = x)$.

d) Show that for constants a, b, and c, $aX + bY + c$ has a normal distribution. Find its mean and variance in terms of the parameters of X and Y.

e) Show that if $\mu_X = \mu_Y = 0$, then $X\cos\theta + Y\sin\theta$ and $-X\sin\theta + Y\cos\theta$ are independent normal variables, where

$$\theta = \frac{1}{2}\cot^{-1}\left[\frac{{\sigma_X}^2 - {\sigma_Y}^2}{2\rho\sigma_X\sigma_Y}\right]$$

Explain the geometric significance of θ in terms of the axes of an ellipse of constant density for (X, Y).

8. Let X_1 and X_2 be two independent standard normal random variables. Define two new random variables as follows: $Y_1 = X_1 + X_2$ and $Y_2 = \alpha X_1 + 2X_2$. You are not given the constant α but it is known that $Cov\{Y_1, Y_2\} = 0$. Find

a) the density of Y_2; b) $Cov\{X_2, Y_2\}$.

9. Suppose that W has normal (μ, σ^2) distribution. Given that $W = w$, suppose that Z has normal $(a\,w + b, \tau^2)$ distribution.

a) Show the joint distribution of W and Z is bivariate normal, and find its parameters.

b) What is the distribution of Z?

c) What is the conditional distribution of W given $Z = z$?

10. Show that if V and W have a bivariate normal distribution then

a) every linear combination $aV + bW$ has a normal distribution;

b) every pair of linear combinations $(aV + bW,\ cV + dW)$ has a bivariate normal distribution.

c) Find the parameters of the distributions obtained in a) and b) in terms of the parameters of the joint distribution of V and W.

11. Show that for standard bivariate normal variables X and Y with correlation ρ,

$$E(\max(X, Y)) = \sqrt{\frac{1-\rho}{\pi}}$$

12. Suppose that the magnitude of a signal received from a satellite is

$$S = a + bV + W$$

where V is a voltage which the satellite is measuring, a and b are constants, and W is a noise term. Suppose V and W are independent and normally distributed with means 0 and variances ${\sigma_V}^2$ and ${\sigma_W}^2$.

a) Find $Corr(S, V)$.

b) Given that $S = s$, what is the distribution of V?

c) What is the best estimate of V given $S = s$?

d) If this estimate is used repeatedly for different values of S coming from a sequence of independent values of V and W with the given normal distributions, what is the long-run average absolute value of the error of estimation?

13. Find a formula in terms of ρ for the ratio of the lengths of the axes of an ellipse of constant density in the standard bivariate normal distribution with correlation ρ. (Let the ratio be the length of the axis at $+45°$ over the length of the axis at $-45°$.) Check your answer by measurement with a ruler in Figure 3 in the case where $\rho = 1/2$. [*Hint:* Let $\rho = \cos\theta$ and reason from Figure 3 that an ellipse of constant density is the image in the (X, Y) plane of the unit circle in the (X, Z) plane. Now consider the images of the points $(\cos\theta/2, \sin\theta/2)$ and $(\cos(\theta/2 + \pi/2), \sin(\theta/2 + \pi/2))$ in the (X, Y) plane which end up on the $\pm 45°$ lines in the (X, Z) plane, and use trigonometric identities.]