The properties of the standard bivariate normal distribution stated in the box on page 451 all follow from the basic representation

$$Y = \rho X + \sqrt{1-\rho^2}Z \qquad (1)$$

in terms of independent standard normal X and Z.

Conditionals. The formula for the distribution of Y given $X = x$ is immediate from (1). Conditioning on X does not affect the distribution of Z. And given $X = x$ you can treat X in (1) as the constant x, so Y is then just a linear transformation of the standard normal variable Z with coefficients involving ρ and x. This gives the conditional distribution of Y given $X = x$. The distribution of X given $Y = y$ follows by symmetry, or from (1′) below.

Symmetry. The standard bivariate normal distribution of (X, Y) is symmetric with respect to switching X and Y. This can be seen from the formula for the joint density, which is a symmetric function of x and y, or from the geometric description of X and Y. This symmetry is obscured in formula (1) however. You should check as an exercise that (1) has a dual

$$X = \rho Y + \sqrt{1-\rho^2}\,Z' \qquad (1')$$

where Z' is a linear combination of X and Z that is independent of Y.

Joint density. The derivation of this is an exercise: Write out the formulae for the marginal and conditional densities, multiply, and simplify. There is no point remembering this formula. Rather, take the following:

Advice. Do not attempt to compute bivariate normal probabilities or expectations by integrating against the joint density. It is always simpler to rewrite the problem in terms of independent variables X and Z, using (1). This technique is used in all the examples below.

Bivariate Normal Distribution

Random variables U and V have *bivariate normal distribution* with parameters μ_U, μ_V, σ_U^2, σ_V^2, and ρ if and only if the standardized variables

$$X = (U - \mu_U)/\sigma_U \qquad Y = (V - \mu_V)/\sigma_V$$

have standard bivariate normal distribution with correlation ρ. Then

$$\rho = Corr(X, Y) = Corr(U, V)$$

and U and V are independent if and only if $\rho = 0$.

Examples

The point of the following examples is to show how any problem involving random variables U and V with a bivariate normal distribution can be solved by a simple three-step procedure:

- **Step 1.** Express U and V in terms of the standardized variables X and Y.
- **Step 2.** Write $Y = \rho X + \sqrt{1-\rho^2}Z$ to reduce the problem to one involving two independent standard normal variables X and Z.
- **Step 3.** Solve the reduced problem involving X and Z by exploiting independence or rotational symmetry.

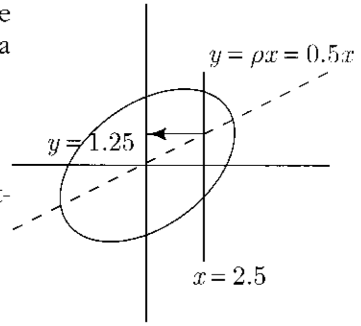
Example 1. **Fathers and sons.**

Galton's student Karl Pearson carried out a study on the resemblances between parents and children. He measured the heights of 1078 fathers and sons, and found that the sons averaged one inch taller than the fathers:

Fathers:	mean height: 5′9″	SD: 2″
Sons:	mean height: 5′10″	SD: 2″
	correlation: 0.5	

Problem 1. Predict the height of the son of a father who is 6′2″ tall.

Solution. Assume that the data are approximately bivariate normal in distribution. Then the parameters can be estimated by the corresponding empirical measurements.



Let X be the father's height in standard units, and Y be the son's height in standard units. The assumption of a bivariate normal distribution makes

$$Y = \rho X + \sqrt{1-\rho^2}Z$$

where Z is standard normal independent of X. The natural prediction for Y given $X = x$ is

$$E(Y|X = x) = \rho x$$

Here the given value of X is

$$\begin{aligned} x &= 6'2'' \text{ converted to standard units} \\ &= (6'2'' - 5'9'')/2'' = 2.5 \text{ standard units} \end{aligned}$$

So the predicted value of Y is

$$E(Y|X = x) = 0.5 \times 2.5 = 1.25 \text{ standard units},$$