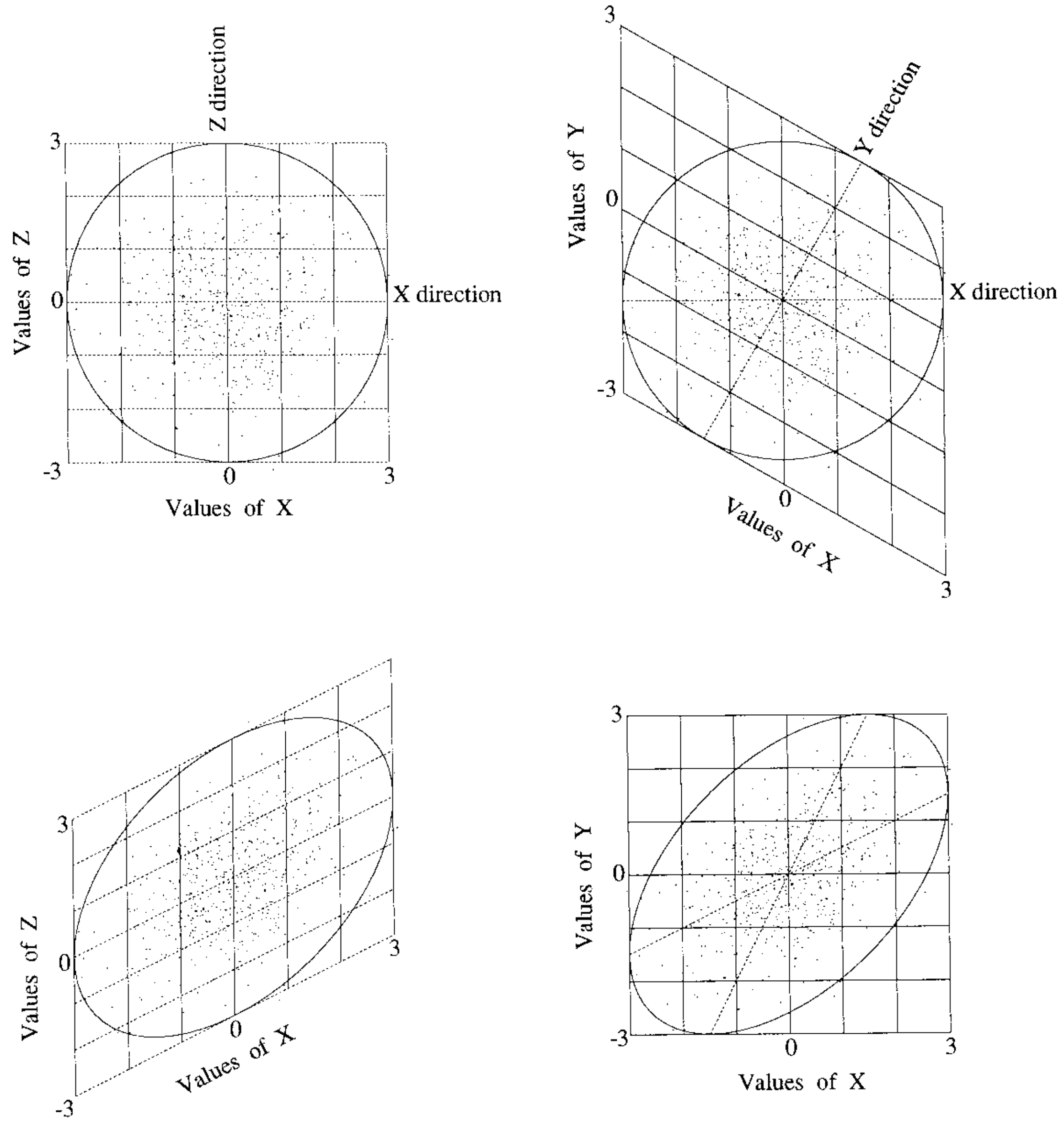
FIGURE 3. **Geometry of the bivariate normal distribution.** Properties of the standard bivariate normal distribution with correlation ρ may be understood in terms of the simplest case $\rho = 0$ by the geometry of the linear transformation $(X, Z) \mapsto (X, Y)$, displayed here for $\theta = 60°$, so

$$\rho = \cos\theta = \frac{1}{2}, \qquad \sqrt{1-\rho^2} = \sin\theta = \frac{\sqrt{3}}{2} \quad \text{and} \quad Y = \frac{1}{2}X + \frac{\sqrt{3}}{2}Z.$$



Key to Figure 3.

Top left panel. This shows a computer-generated scatter of 500 points picked at random according to the joint distribution of X and Z, plotted in the usual way with rectangular X and Z coordinates. This is a roughly circular cloud, due to the rotational symmetry of the distribution of two independent standard normals. The circle is the contour of constant density for (X, Z), of radius 3 standard units, containing 98.9% of the probability. The vertical lines represent the events $X = 0, \pm1, \pm2, \pm3$. The dashed horizontal lines represent $Z = 0, \pm1, \pm2, \pm3$.

Top right panel. This is the same scatter in the (X, Z) plane, but with the diagonal lines $Y = 0, \pm1, \pm2, \pm3$. The Y direction is the dotted line at angle $\theta = 60°$ to the horizontal X direction. The diagonals $Y = constant$ are at angle θ to the vertical lines $X = constant$.

Bottom right panel. This is the image of the top right panel after shearing and shrinking to represent X and Y by new rectangular axes. Each point in the top scatter is transformed into one in the bottom scatter. Thus the cloud becomes a random scatter of 500 points picked at random according to the bivariate normal distribution of X and Y, with correlation $\rho = \cos\theta$. Think of the lines in the top right panel as a lattice of rigid rods attached by pins. Keep the vertical axis $X = 0$ fixed, and shear the lattice so the diagonals become horizontal. This makes a lattice of squares of side $1/\sin\theta$. Now shrink everything by a factor of $\sin\theta$ to get the bottom-right panel.

The shearing which turns the diamonds into squares turns the circle into an ellipse, with major axis on the 45-degree line through the new origin. This is an ellipse of constant density for (X, Y). The images of the dotted lines in the old X and Y directions are the dotted lines $Y = \rho X$ and $X = \rho Y$. These are the *regression lines* discussed further in the next paragraph.

Bottom left panel. This is the image of the top left panel by the same transformation from (X, Z) to (X, Y). The ellipse and the cloud of points are the same as in the bottom right panel. But now the lines representing $X = 0, \pm1, \pm2, \pm3$ are shown, along with those representing $Z = 0, \pm1, \pm2, \pm3$. The line $Z = 0$ plays a particularly important role. This is the *regression line*. The equation of this line $Z = 0$ in the (X, Y) plane is

$$Y = \rho X$$

where ρ is the correlation. Geometrically, this is the line of midpoints of vertical sections of the ellipse. Statistically, it is the best predictor of Y based on X.