Let \((X_i, Y_i)\) be independent \(N(\alpha_i, \sigma^2)\) for \(i = 1, \ldots, n\). The MLE for \(\alpha_i\) is \(\hat{\alpha}_i = (X_i + Y_i)/2\). The MLE for \(\sigma^2\) is \(\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} s_i^2\), where \(s_i^2 = [(X_i - \hat{\alpha}_i)^2 + (Y_i - \hat{\alpha}_i)^2]/2 = (X_i - Y_i)^2/4\), because \(X_i - \hat{\alpha}_i = (X_i - Y_i)/2\) and \(Y_i - \hat{\alpha}_i = (Y_i - X_i)/2\). So \(E(s_i^2) = \sigma^2/2\) and the MLE is inconsistent as \(n \to \infty\). This is a “fixed-effects” model with two observations on each effect \(\alpha_i\). The effect is estimated by the mean of the relevant observations: with only two observations per parameter, \(\hat{\alpha}_i\) remains quite variable as \(n \to \infty\). The common variance \(\sigma^2\) is estimated by the mean of the sample variances, with the sample size as the divisor, rather than degrees of freedom. The number of observations relevant to estimating \(\sigma^2\) grows without bound, but inconsistency follows from the bias in the MLE.

To verify the formulas for the MLE, set \(v = 1/\sigma^2\): this makes the calculus a little easier. The log likelihood is

\[
2n \log \frac{1}{\sqrt{2\pi}} + n \log v - v \sum_{i=1}^{n} \frac{(X_i - \alpha_i)^2 + (Y_i - \alpha_i)^2}{2}
\]

It’s “obvious” that \(\hat{\alpha}_i\) is as claimed. At these values for \(\alpha_i\), the derivative with respect to \(v\) is

\[
\frac{n}{v} - \sum_{i=1}^{n} s_i^2
\]

so

\[
\hat{v} = \frac{n}{\sum_{i=1}^{n} s_i^2}
\]

and

\[
\hat{\sigma}^2 = \frac{1}{\hat{v}} = \frac{1}{n} \sum_{i=1}^{n} s_i^2
\]

as required.