The Empirical FT.

Data $\{X(0), X(1), \ldots X(T-1)\}$

$$EFT: d_X^T(\lambda) = \Sigma_1^N \Sigma X(t) \exp\{-i\lambda t\} = \int_0^T \quad \lambda \text{ real}$$

$$\text{Note } d_N^T(0) = \Sigma X(t)$$

What is the large sample distribution of the EFT?

The complex normal.

The complex normal, $N^C(\mu,\sigma^2)$, is a variate of the form

$$Y = U + i\,V$$

where U and V are independent $N(\mathrm{Re}\,\mu, \sigma^2/2)$, $N(\mathrm{Im}\,\mu, \sigma^2/2)$

Notes:

$$EY = \mu$$

$$\mathrm{var}\,Y = E\,|Y-\mu|^2 = \sigma^2$$

$$N^C(0,1) = IN(0,1/2) = (Z_1 + iZ_2)/\sqrt{2}$$

$$|N^C(0,1)|^2 = (Z_1^2 + Z_2^2)/2 = \chi_2^2/2 = \text{exponential}$$

$$\chi_\nu^2 = Z_1^2 + \ldots + Z_\nu^2 \qquad Z_j\;\; IN(0,1)$$

$$E\chi_\nu^2 = \nu \qquad \mathrm{var}\,\chi_\nu^2 = 2\nu$$

Theorem. Suppose X is stationary mixing, then

$i). \, d_X^T(\lambda)$ is asymptotically

$$N(Tp_X, 2\pi T f_{XX}(0)), \quad \lambda = 0$$

$$N^C(0, 2\pi T f_{XX}(\lambda)), \quad \lambda \neq 0$$

$ii). \, d_X^T(\lambda_1), \ldots, d_X^T(\lambda_L), \lambda_1, \ldots, \lambda_L$ distinct and $\neq 0$ are asymp

$$IN^C(0, 2\pi T f_{XX}(\lambda_l))$$

$iii). \, d^T(\frac{2\pi r_1}{T}), \ldots, d^T(\frac{2\pi r_L}{T}), r_1, \ldots, r_L$ distinct integers $\neq 0$ with $2\pi r_l / T \sim \lambda$

are asymp $IN^C(0, 2\pi T f_{XX}(\lambda))$

Proof. Write

$$d_X^T(\lambda) = \int \Delta^T(\lambda - \alpha)\, dZ_X(\alpha)$$

Evaluate first and second-order cumulants

Bound higher cumulants

Normal is determined by its moments

Consider

$$\operatorname{cov}\{d_X^T(\lambda), d_X^T(\mu)\} = \iint \Delta^T(\lambda-\alpha)\Delta^T(-\mu+\beta) f_{NN}(\alpha)\delta(\lambda-\alpha)d\alpha d\beta$$

$$= \int \Delta^T(\lambda-\alpha)\Delta^T(-\mu+\alpha) f_{XX}(\alpha)d\alpha$$

$$\sim 2\pi\Delta^T(\lambda-\mu) f_{XX}(\lambda)$$

We have

$$\int \Delta^T(\lambda-\alpha)\overline{\Delta^T(\mu-\alpha)}d\alpha = 2\pi\Delta^T(\lambda-\mu)$$

Comments.

Already used to study rate estimate

Tapering makes

$$\operatorname{var} d_X^T(\lambda) \sim \int | H^T(\lambda-\alpha) | f_{XX}(\alpha) d\alpha$$

Get asymp independence for different frequencies

The frequencies $2\pi r/T$ are special, e.g. $\Delta^T(2\pi r/T)=0,\ r\neq 0$

Also get asymp independence if consider separate stretches

p-vector version involves p by p spectral density matrix $f_{XX}(\lambda)$

Estimation of the (power) spectrum.

For $\lambda \neq 0$, consider the periodogram,

$$I_{XX}^{T}(\lambda) = \frac{1}{2\pi T} | d_{X}^{T}(\lambda) |^{2}$$

$$\sim \frac{1}{2\pi T} | N^{C}(0, 2\pi T f_{XX}(\lambda)) |^{2}$$

$$\sim f_{XX}(\lambda)\chi_{2}^{2}/2, \quad \text{exponential}$$

Estimate appears inconsistent unless $f_{XX}(\lambda) = 0$

but note $E\{ f_{XX}(\lambda)\chi_{2}^{2}/2\} = f_{XX}(\lambda)$

and $\mathrm{var}\{ f_{XX}(\lambda)\chi_{2}^{2}/2\} = f_{XX}(\lambda)^{2}$

An estimate whose limit is a random variable

Some moments.

$$E\,|\,\zeta\,|^2 = \operatorname{var}\zeta + |\,E\zeta\,|^2$$

$$Ed_X^T(\lambda) = \int_0^T \Sigma X(t)\exp\{-i\lambda t\} = c_X \Delta^T(\lambda)$$

so

$$E\,|\,d_X^T(\lambda)\,|^2 = \int |\,\Delta^T(\lambda-\alpha)\,|^2\; f_{XX}(\alpha)d\alpha + p_N^2\;|\,\delta^T(\lambda)\,|^2$$

The estimate is asymptotically unbiased

Final term drops out if $\lambda = 2\pi r/T \neq 0$

Best to correct for mean, work with

$$d_X^T(\lambda) - c_X^T \Delta^T(\lambda)$$

Periodogram values are asymptotically independent since d^T values are -

independent exponentials

Use to form estimates

Smoothed periodogram. Consider $r_1, \ldots, r_L$ distinct integers $\neq 0$

$2\pi r_l / T$ near λ

Estimate

$$f_{XX}^T(\lambda) = \sum_l I^T(2\pi r_l / T) / L$$

CLT gives

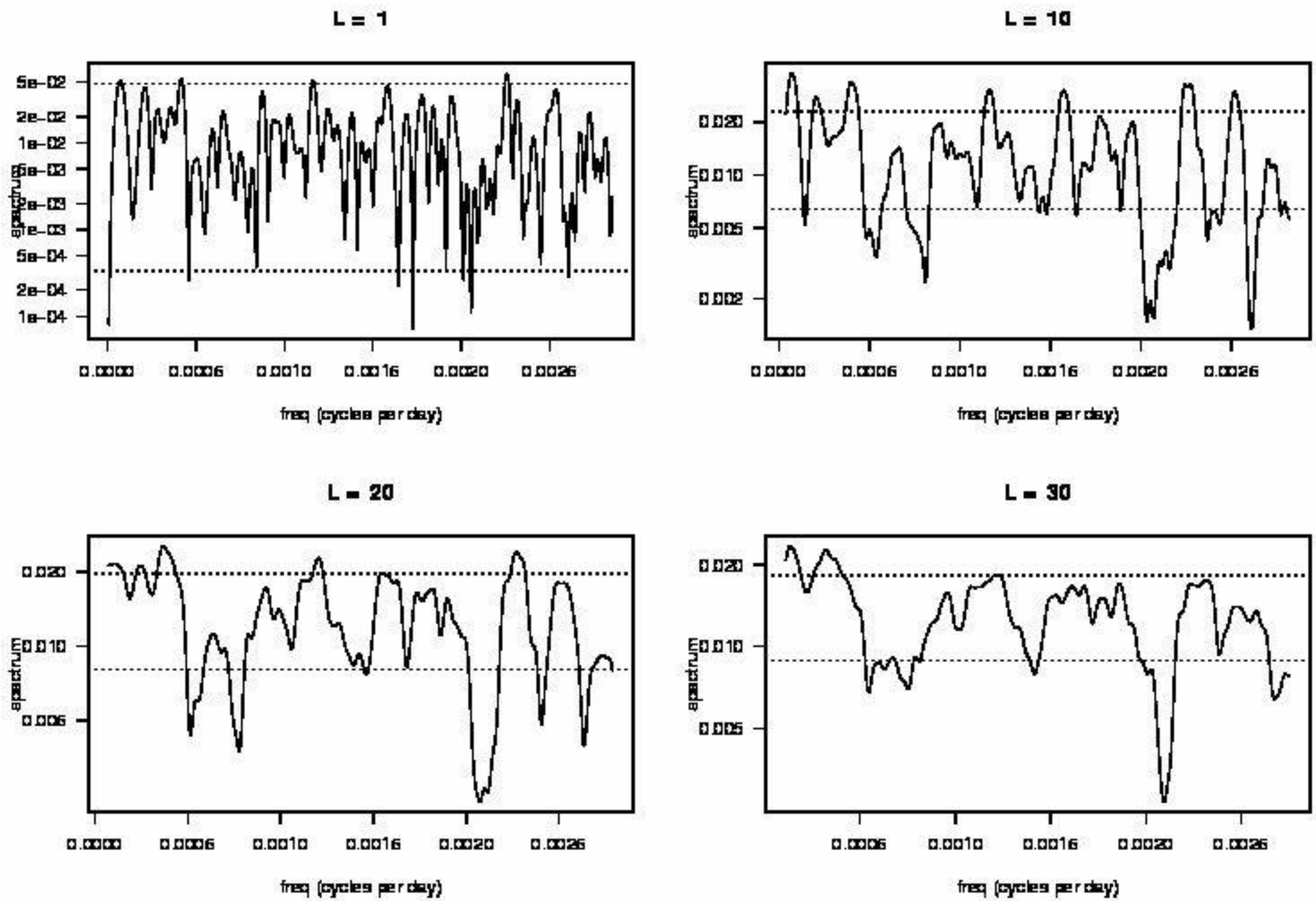
$$f_{XX}^T(\lambda) \rightarrow f_{XX}(\lambda)\chi_{2L}^2 / 2L \text{ in distribution}$$

Now

$$E\{f_{XX}(\lambda)\chi_{2L}^2 / 2L\} = f_{XX}(\lambda) \qquad \text{var}\{f_{NN}(\lambda)\chi_{2L}^2 / 2L\} = f_{XX}(\lambda)^2 / L$$

Can control variance. Try several L's

California eathquakes 1932–1992



Approximate marginal confidence intervals

$$\Pr\{\log f^T(\lambda) + \log \upsilon / \chi_\upsilon^2(1-\alpha/2) < \log f(\lambda) < \log f^T(\lambda) + \log \upsilon / \chi_\upsilon^2(\alpha/2)\}$$

Notes.

variance stabilized by log

set CI about mean level

simultaneous band via extreme value distribution

might take $L \to \infty$ for consistent estimate

might take weighted mean, or tapered FT

might split data, $T = LV$

More on choice of L

Consider

$$E\{f^{\mathrm{T}}(\lambda)\} = E\{\sum_l I^T(\lambda_l)/L\}, \quad \lambda_l = 2\pi r_l/T \approx \lambda = 2\pi r/T$$

$$= \int \frac{1}{L}\sum_l \frac{1}{2\pi T}\left[\sin T(\lambda_l - \alpha)/2)/((\lambda_l - \alpha)/2)\right]^2 f(\alpha)d\alpha$$

$$= \int \mathrm{W}^{\mathrm{T}}(\lambda - \alpha) f(\alpha)d\alpha$$

Choice of L affects width of $W^T(.)$

$L2\pi/T$ radians

If f is not constant, f^T biased

Approximation to bias

Suppose $W^T(\boldsymbol{\alpha}) = B_T^{-1} W(B_T^{-1}\boldsymbol{\alpha})$

$$
\begin{aligned}
&\int W^T(\boldsymbol{\alpha}) f(\boldsymbol{\lambda} - \boldsymbol{\alpha}) d\boldsymbol{\alpha} \\
&\quad = \int W(\boldsymbol{\beta}) f(\boldsymbol{\lambda} - B_T \boldsymbol{\beta}) \\
&\quad = \int W(\boldsymbol{\beta}) [f(\boldsymbol{\lambda}) - B_T \boldsymbol{\beta} f'(\boldsymbol{\lambda}) + B_T^2 \boldsymbol{\beta}^2 f''(\boldsymbol{\lambda})/2 + \ldots] d\boldsymbol{\beta} \\
&\quad = f(\boldsymbol{\lambda}) \int W(\boldsymbol{\beta}) d\boldsymbol{\beta} - B_T f'(\boldsymbol{\lambda}) \int \boldsymbol{\beta} W(\boldsymbol{\beta}) d\boldsymbol{\beta} + B_T^2 f''(\boldsymbol{\lambda}) \int \boldsymbol{\beta}_T^2 W(\boldsymbol{\beta}) d\boldsymbol{\beta}/2 + \ldots
\end{aligned}
$$

For W symmetric $\int \boldsymbol{\beta} W(\boldsymbol{\beta}) d\boldsymbol{\beta} = 0$

Indirect estimate

$$\frac{1}{2\pi} \int \Sigma \exp\{-i\lambda u\} w^T(u) c^T_{xx}(u) du$$

Estimation of finite dimensional θ.

approximate likelihood (assuming I^T values independent exponentials)

spectrum $f(\lambda;\theta)$, θ in Θ

$$L(\theta) = \prod_r f(2\pi r/T)^{-1} \exp\{-I^T(2\pi r/T)/f(2\pi r/T)\}$$

Bivariate case.

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \int \exp\{it\lambda\} \begin{bmatrix} dZ_X(\lambda) \\ dZ_Y(\lambda) \end{bmatrix}$$

$$\operatorname{cov}\{dZ_X(\lambda), dZ_Y(\mu)\} = \delta(\lambda - \mu) f_{XY}(\lambda) d\mu$$

spectral density matrix

$$\begin{bmatrix} f_{XX}(\lambda) & f_{XY}(\lambda) \\ f_{YX}(\lambda) & f_{YY}(\lambda) \end{bmatrix}$$

Crossperiodogram.

$$I_{XY}^{T}(\lambda) = \frac{1}{2\pi T} d_{X}^{T}(\lambda)\overline{d_{Y}^{T}(\lambda)}$$

matrix form $\mathbf{I}^{T}(\lambda) = \begin{bmatrix} I_{XX}^{T} & I_{XY}^{T} \\ I_{YX}^{T} & I_{YY}^{T} \end{bmatrix}$

Smoothed periodogram.

$$\mathbf{f}_{NN}^{T}(\lambda) = \sum_{l} \mathbf{I}^{T}(2\pi r_l / T) / L, \quad 2\pi r_l / T \approx \lambda, l = 1,\ldots, L$$

Complex Wishart

$$\mathbf{X}_1, \ldots, \mathbf{X}_n \sim IN_r(\mathbf{0}, \Sigma)$$

$$W_r^C(n, \mathbf{\Sigma}) \sim \mathbf{W} = \sum_1^n \mathbf{X}_j \overline{\mathbf{X}_j^T}$$

diagonals chi-squared

$$E\mathbf{W} = n\mathbf{\Sigma}$$

Predicting Y via X

predicting $dZ_Y(\lambda)$ by $dZ_X(\lambda)$

$$\min_A E\,|\,dZ_Y(\lambda) - A\,dZ_X(\lambda)\,|^2$$

$$A(\lambda) = f_{YX}(\lambda)\, f_{XX}(\lambda)^{-1}, \quad \text{transfer function}$$

$$|A(\lambda)|: \text{ gain} \quad \arg A(\lambda): \text{ phase}$$

$$\text{MSE}: [1-|R(\lambda)|^2]\, f_{YY}(\lambda)$$

$$\text{coherency}: R(\lambda) = f_{YX}(\lambda) / \sqrt{f_{XX}(\lambda)\, f_{YY}(\lambda)}$$

$$a(u) = (2\pi)^{-1} \int A(\alpha) \exp\{iu\alpha\}\, d\alpha$$

$$X(t) \approx \int \exp\{i\lambda t\} A(\lambda)\, dZ_X(\lambda)$$

Plug in estimates.

$$A^T(\lambda) = f_{YX}^T(\lambda) f_{XX}^T(\lambda)^{-1}$$

$$|A^T(\lambda)| \quad \arg A^T(\lambda)$$

$$\text{MSE}: [1 - |R^T(\lambda)|^2] f_{YY}^T(\lambda)$$

$$R^T(\lambda) = f_{YX}^T(\lambda) / \sqrt{f_{XX}^T(\lambda) f_{YY}^T(\lambda)}$$

Density of $|R^T|^2$

$$(1 - |R|^2)^L {}_2F_1(L, L; 1; |R|^2 |R^T|^2) \frac{\Gamma(L)}{\Gamma(L-1)\Gamma(1)}$$

If $|R|^2 = 0$ approx $100\alpha\%$ point

$$1 - (1 - \alpha)^{1/(L-1)}$$

$$E|R^T|^2 \approx 1/L$$

Large sample distributions.

$$\mathrm{var}\ \log|A^T| \propto [|R|^{-2} - 1]/L$$

$$\mathrm{var}\ \arg A^T \propto [|R|^{-2} - 1]/L$$

Advantages of frequency domain approach.

techniques for many stationary processes look the same

approximate i.i.d sample values

assessing models (character of departure)

time varying variant

...

Networks.

partial spectra - trivariate (M,N,O)

Remove linear time invariant effect of M from N and O and examine coherency of residuals,

$$dZ_{N|M}(\lambda) = dZ_N(\lambda) - A(\lambda)dZ_M(\lambda), \quad A = f_{NM} f_{MM}^{-1}$$

$$dZ_{O|M}(\lambda) = dZ_O(\lambda) - B(\lambda)dZ_M(\lambda), \quad B = f_{OM} f_{MM}^{-1}$$

partial cross-spectrum of N and O having removed linear time invariant effects of M

$$f_{NO|M}(\lambda) = f_{NO}(\lambda) - f_{NM}(\lambda) f_{MM}(\lambda)^{-1} f_{MN}(\lambda)$$

$$R^T_{NO|M}(\lambda) =$$

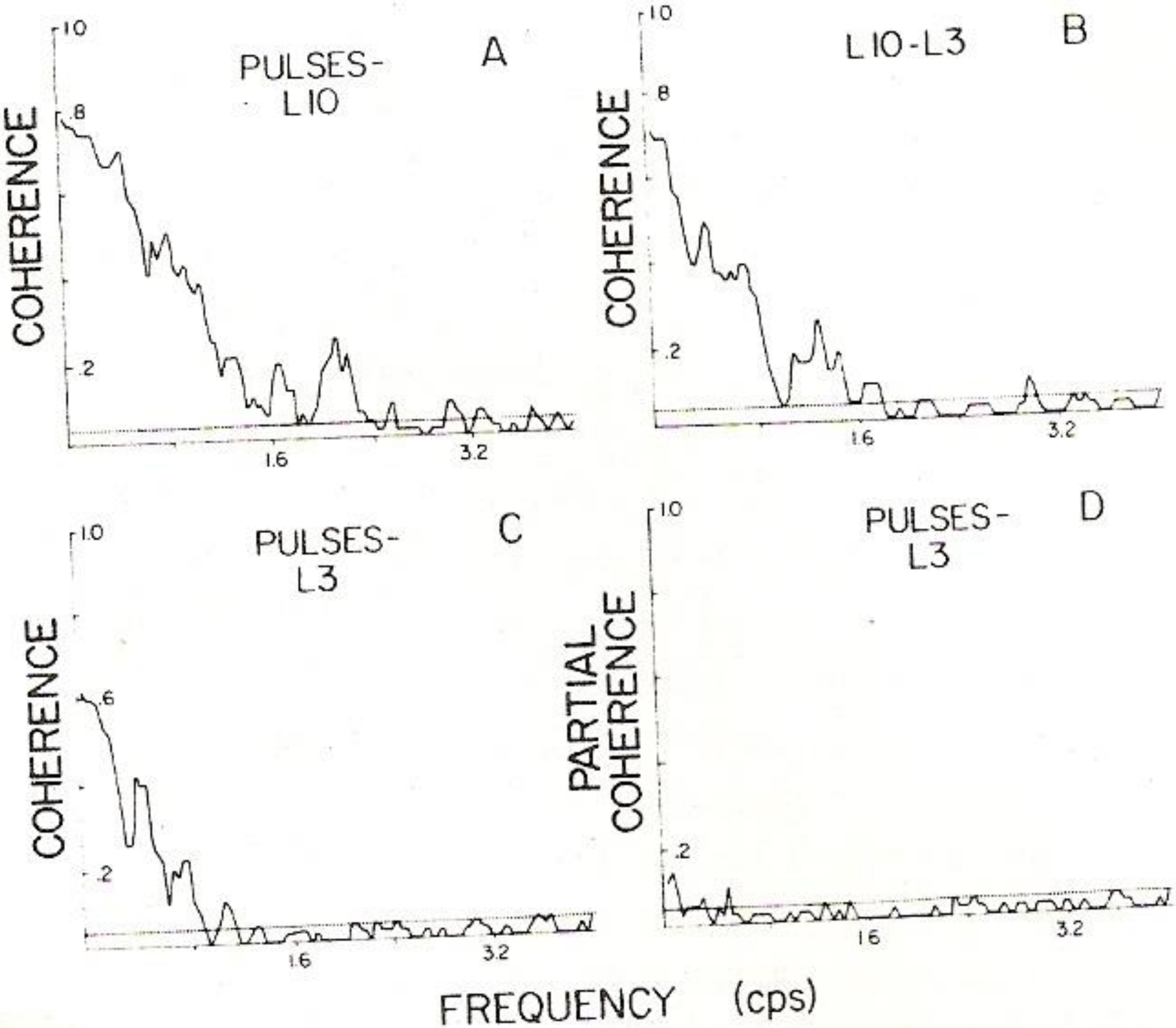
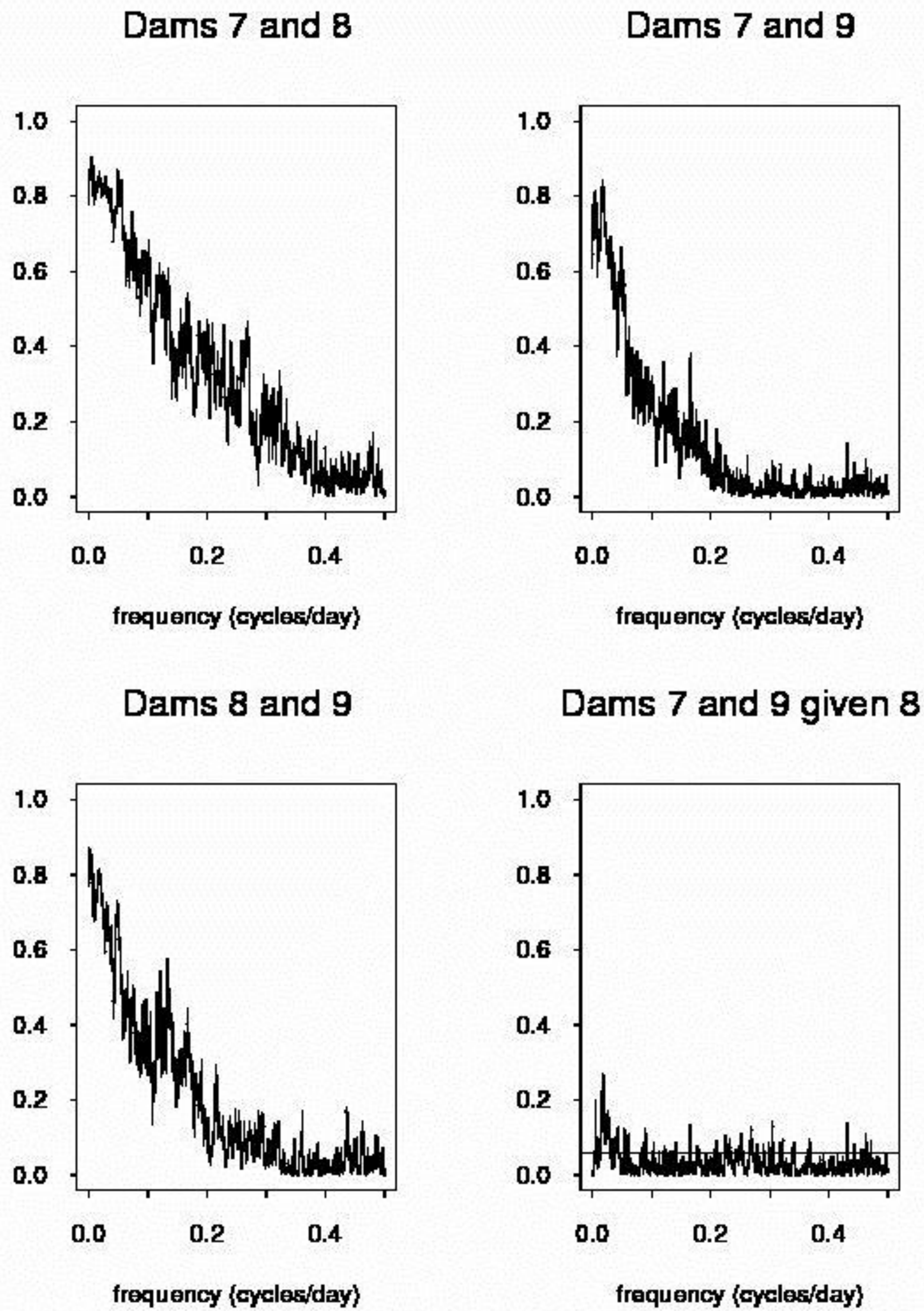


Fig. 5. Value of partial coherences. Confirmation of the a priori



Dams 7 and 8
Dams 7 and 9
Dams 8 and 9
Dams 7 and 9 given 8
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
frequency (cycles/day)

Point process system.

An operation carrying a point process, M, into another, N

$N = S[M]$

$S = \{\text{input,mapping, output}\}$

$Pr\{dN(t)=1|M\} = \{\mu + \int a(t-u)dM(u)\}dt$

System identification: determining the charcteristics of a system from inputs and corresponding outputs

$\{M(t),N(t);0\leq t<T\}$

Like regression vs. bivariate

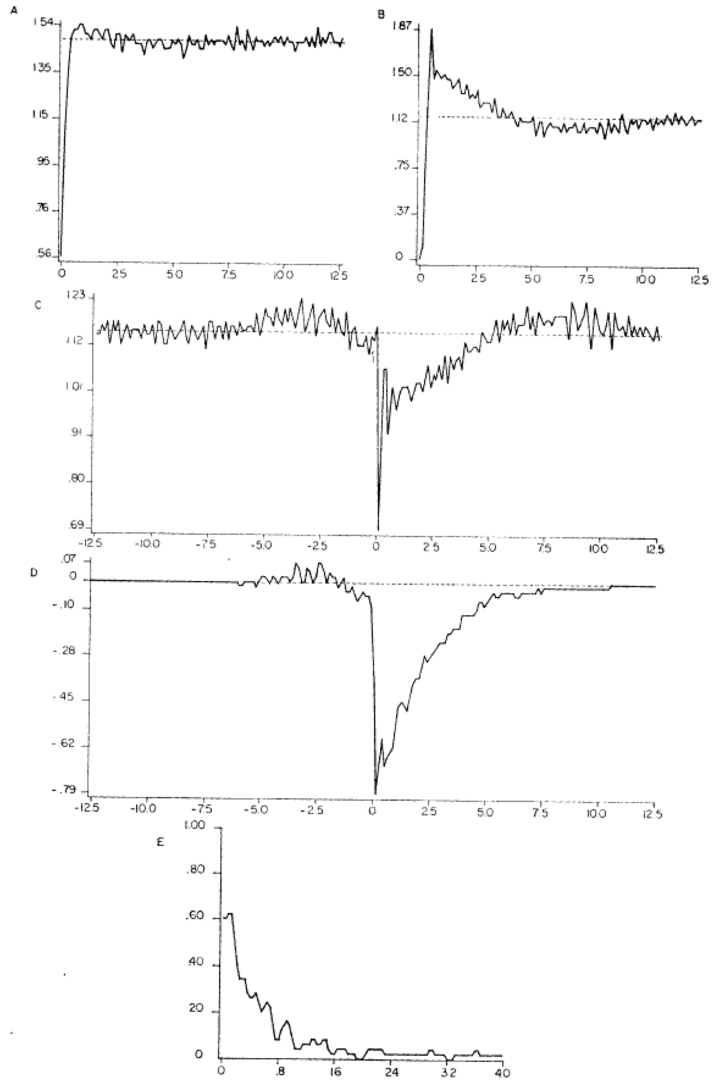

A
1.54
1.35
1.15
.95
.76
.56
0
25
50
75
100
125
B
1.87
1.50
1.12
.75
.37
0
C
1.23
1.12
1.01
.91
.80
.69
-12.5
-10.0
-7.5
-5.0
-2.5
2.5
5.0
7.5
10.0
12.5
D
.07
-.10
-.28
-.45
-.62
-.79
E
1.00
.80
.60
.40
20
.8
16
24
32
40

FIG. 2.

Realizable case.

$a(u) = 0 \quad u<0$

$A(\lambda)$ is of special form

e.g. $N(t) = M(t-\tau)$

$\arg A(\lambda) = -\lambda\tau$

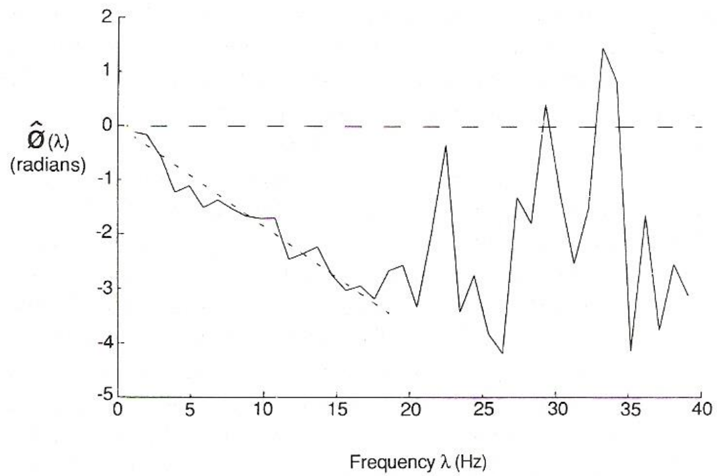


FIG. 4. Estimated phase, $\hat{\Phi}(\lambda)$, for the response of a muscle spindle II ending to stimulation of a static fusimotor axon. The coherence for the same relation is illustrated in Fig. 3d. The dashed line represents the linear regression line fitted to the phase curve over the range of frequencies where the coherence shown in Fig. 3d is significant. The slope of the linear phase curve gives an estimated delay of 29.9 msec with 95% confidence limits of 29.9 ± 2.18 msec.