Time series power spectral density.

frequency-side, λ, vs. time-side, t

$X(t)$, $t = 0, \pm 1, \pm 2, \ldots$ Suppose stationary

$c_{XX}(u) = \text{cov}\{X(t+u), X(t)\}$ $\quad u = 0, \pm 1, \pm 2, \ldots$ lag

$f_{XX}(\lambda) = (1/2\pi)\, \Sigma \exp\{-i\lambda u\}\, c_{XX}(u) \qquad -\pi < \lambda \leq \pi$

period 2π non-negative

$\lambda/2\pi$: *frequency* (cycles/unit time)

$$c_{XX}(u) = \int_{-pi}^{pi} \exp\{iu\lambda\}\, f_{XX}(\lambda)\, d\lambda$$

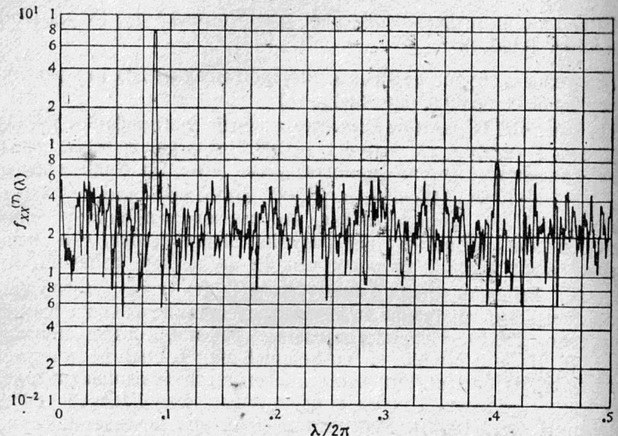


Рис. 5.4.5. Оценка $f_{XX}^{(T)}(\lambda)$ составного ряда осадков для Англии и Уэльса за 1789—1959 гг. с осреднением пяти ординат периодограммы (логарифмический масштаб). (По горизонтали — частоты в цикл/месяц.)

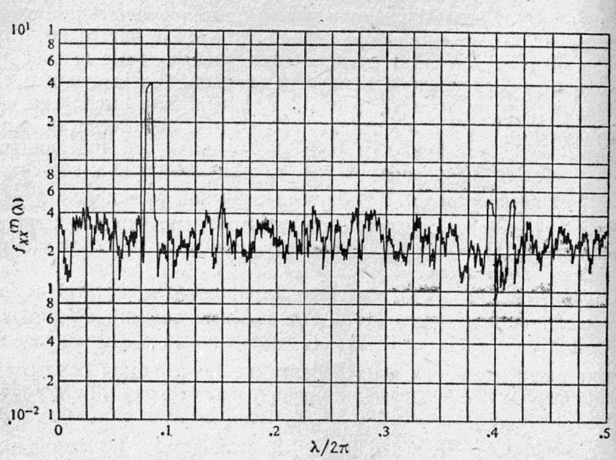


Рис. 5.4.6. Оценка $f_{XX}^{(T)}(\lambda)$ составного ряда осадков для Англии и Уэльса за 1789—1959 гг. с осреднением одиннадцати ординат периодограммы (логарифмический масштаб). (По горизонтали — частоты в цикл/месяц.)

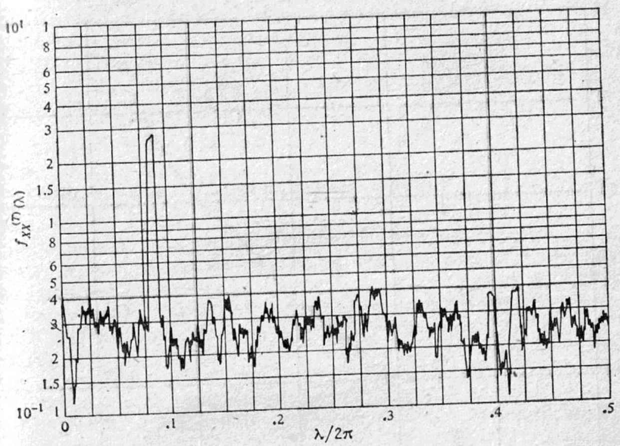


Рис. 5.4.7. Оценка $f_{XX}^{(T)}(\lambda)$ составного ряда осадков для Англии и Уэльса за 1789—1959 гг. с осреднением пятнадцати ординат периодограммы (логарифмический масштаб.) (По горизонтали — частоты в цикл/месяц.)

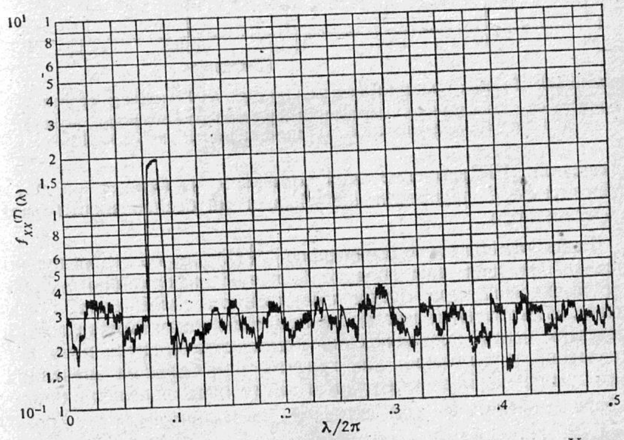


Рис. 5.4.8. Оценка $f_{XX}^{(T)}(\lambda)$ составного ряда осадков для Англии и Уэльса за 1789—1959 гг. с осреднением двадцати одной ординаты периодограммы (логарифмический масштаб). (По горизонтали — частоты в цикл/месяц.)

The Empirical FT.

Data $X(0), \ldots, X(T-1)$

$$d_X^T(\lambda) = \Sigma \exp\{-i\lambda u\} X(u)$$

$$d_X^T(0) = \Sigma X(u)$$

What is the large sample distribution of the EFT?

The complex normal.

The complex normal, $N^C(\mu,\sigma^2)$, is a variate of the form

$$Y = U + i\,V$$

where U and V are independent $N(\mathrm{Re}\,\mu,\sigma^2/2)$, $N(\mathrm{Im}\,\mu,\sigma^2/2)$

Notes:

$$EY = \mu$$

$$\mathrm{var}\,Y = E\,|Y-\mu|^2 = \sigma^2$$

$$N^C(0,1) = IN(0,1/2) = (Z_1 + iZ_2)/\sqrt{2}$$

$$|N^C(0,1)|^2 = (Z_1^2 + Z_2^2)/2 = \chi_2^2/2 = \text{exponential}$$

$$\chi_\nu^2 = Z_1^2 + \ldots + Z_\nu^2 \qquad Z_j \; IN(0,1)$$

$$E\chi_\nu^2 = \nu \qquad \mathrm{var}\,\chi_\nu^2 = 2\nu$$

Theorem. Suppose X is stationary mixing, then

$i).\ d_X^T(\lambda)$ is asymptotically

$$N(Tc_X, 2\pi T f_{XX}(0)), \quad \lambda = 0$$

$$N^C(0, 2\pi T f_{XX}(\lambda)), \quad \lambda \neq 0$$

$ii).\ d_X^T(\lambda_1),\ldots,d_X^T(\lambda_L),\ \lambda_1,\ldots,\lambda_L$ distinct and $\neq 0$ are asymp

$$IN^C(0, 2\pi T f_{XX}(\lambda_l))$$

$iii).\ d^T(\frac{2\pi r_1}{T}),\ldots,d^T(\frac{2\pi r_L}{T}),\ r_1,\ldots,r_L$ distinct integers $\neq 0$ with $2\pi r_l / T \sim \lambda$

are asymp $IN^C(0, 2\pi T f_{XX}(\lambda))$

Proof. Write

$$d_X^T(\lambda) = \int \Delta^T(\lambda - \alpha) dZ_X(\alpha)$$

Evaluate first and second-order cumulants

Bound higher cumulants

Normal is determined by its moments

Consider

$$\operatorname{cov}\{d_X^T(\lambda), d_X^T(\mu)\} = \int\int \Delta^T(\lambda - \alpha)\Delta^T(-\mu + \beta) f_{XX}(\alpha)\delta(\lambda - \alpha) d\alpha d\beta$$

$$= \int \Delta^T(\lambda - \alpha)\Delta^T(-\mu + \alpha) f_{XX}(\alpha) d\alpha$$

$$\sim 2\pi\Delta^T(\lambda - \mu) f_{XX}(\lambda)$$

We have

$$\int \Delta^T(\lambda - \alpha)\overline{\Delta^T(\mu - \alpha)} d\alpha = 2\pi\Delta^T(\lambda - \mu)$$

Comments.

Already used to study mean estimate

Tapering, h(t)X(t). makes

$$\operatorname{var} d_X^T(\lambda) \sim \int |H^T(\lambda-\alpha)|^2 f_{XX}(\alpha)d\alpha$$

Get asymp independence for different frequencies

The frequencies $2\pi r/T$ are special, e.g. $\Delta^T(2\pi r/T)=0,\ r\neq 0$

Also get asymp independence if consider separate stretches

p-vector version involves p by p spectral density matrix $f_{XX}(\lambda)$

Estimation of the (power) spectrum.

For $\lambda \neq 0$, consider the periodogram,

$$I_{xx}^{T}(\lambda) = \frac{1}{2\pi T} | d_{x}^{T}(\lambda) |^{2}$$

$$\sim \frac{1}{2\pi T} | N^{C}(0, 2\pi T f_{xx}(\lambda)) |^{2}$$

$$\sim f_{xx}(\lambda) \chi_{2}^{2} / 2, \quad \text{exponential}$$

Estimate appears inconsistent unless $f_{xx}(\lambda) = 0$

but note $E\{ f_{xx}(\lambda) \chi_{2}^{2} / 2\} = f_{xx}(\lambda)$

and $\text{var}\{ f_{xx}(\lambda) \chi_{2}^{2} / 2\} = f_{xx}(\lambda)^{2}$

An estimate whose limit is a random variable

Some moments.

$$E|\zeta|^2 = \text{var}\,\zeta + |E\zeta|^2$$

$$Ed_X^T(\lambda) = \int_0^T \Sigma X(t)\exp\{-i\lambda t\} = c_X \Delta^T(\lambda)$$

so

$$E|d_X^T(\lambda)|^2 = \int|\Delta^T(\lambda-\alpha)|^2 f_{XX}(\alpha)d\alpha + c_N^2 |\delta^T(\lambda)|^2$$

The estimate is asymptotically unbiased

Final term drops out if $\lambda = 2\pi r/T \neq 0$

Best to correct for mean, work with

$$d_X^T(\lambda) - c_X^T \Delta^T(\lambda)$$

Periodogram values are asymptotically independent since d^T values are -

independent exponentials

Use to form estimates

Smoothed periodogram. Consider $r_1, \ldots, r_L$ distinct integers $\neq 0$

$2\pi r_l / T$ near λ

Estimate

$$f_{XX}^T(\lambda) = \sum_l I^T(2\pi r_l / T) / L$$

CLT gives

$$f_{XX}^T(\lambda) \to f_{XX}(\lambda)\chi_{2L}^2 / 2L \text{ in distribution}$$

Now

$$E\{f_{XX}(\lambda)\chi_{2L}^2 / 2L\} = f_{XX}(\lambda) \qquad \text{var}\{f_{NN}(\lambda)\chi_{2L}^2 / 2L\} = f_{XX}(\lambda)^2 / L$$

Can control variance. Try several L's

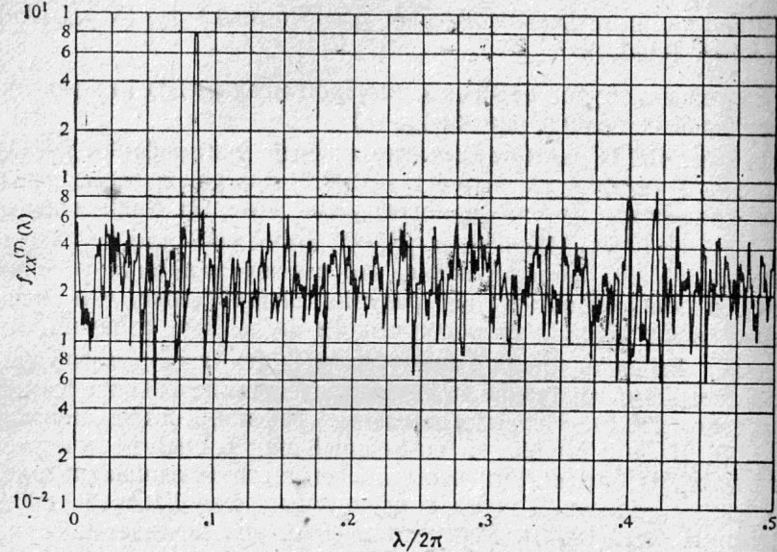


Рис. 5.4.5. Оценка $f_{XX}^{(T)}(\lambda)$ составного ряда осадков для Англии и Уэльса за 1789—1959 гг. с осреднением пяти ординат периодограммы (логарифмический масштаб). (По горизонтали — частоты в цикл/месяц.)

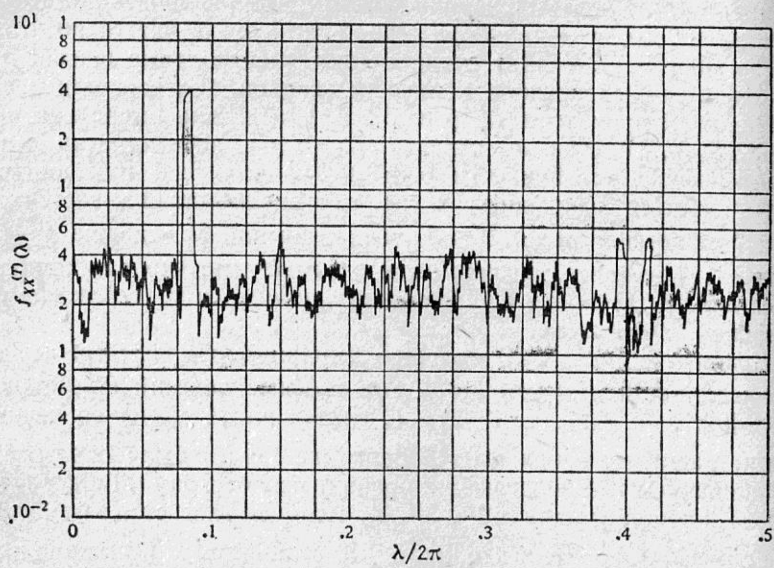


Рис. 5.4.6. Оценка $f_{XX}^{(T)}(\lambda)$ составного ряда осадков для Англии и Уэльса за 1789—1959 гг. с осреднением одиннадцати ординат периодограммы (логарифмический масштаб). (По горизонтали — частоты в цикл/месяц.)

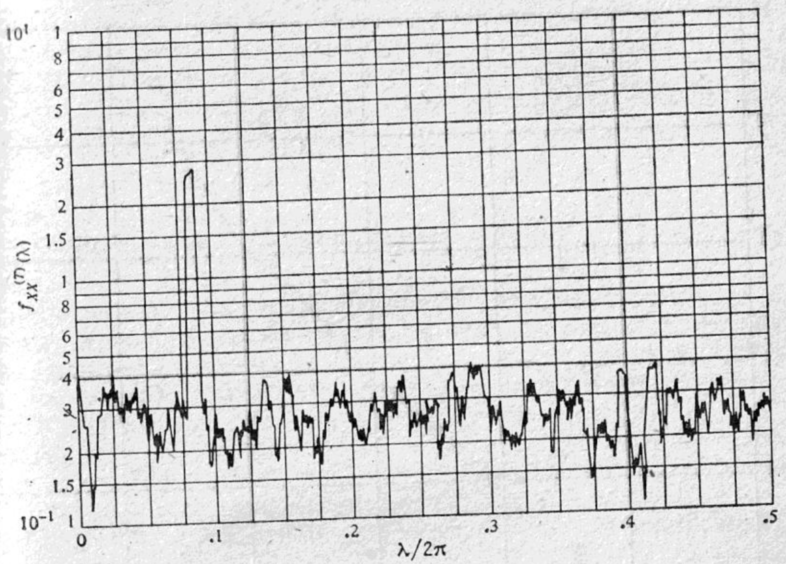


Рис. 5.4.7. Оценка $f_{XX}^{(T)}(\lambda)$ составного ряда осадков для Англии и Уэльса за 1789—1959 гг. с осреднением пятнадцати ординат периодограммы (логарифмический масштаб.) (По горизонтали — частоты в цикл/месяц.)

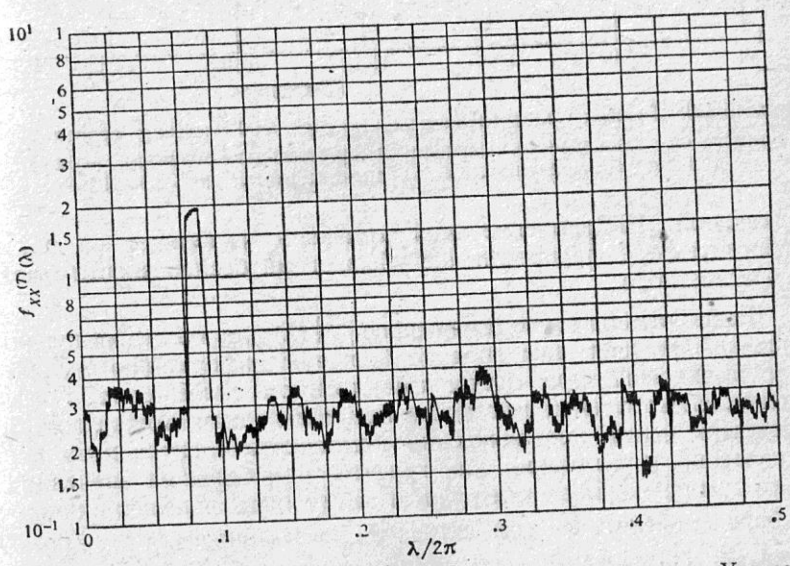


Рис. 5.4.8. Оценка $f_{XX}^{(T)}(\lambda)$ составного ряда осадков для Англии и Уэльса за 1789—1959 гг. с осреднением двадцати одной ординаты периодограммы (логарифмический масштаб). (По горизонтали — частоты в цикл/месяц.)

Approximate marginal confidence intervals

$$\Pr\{\log f^T(\lambda) + \log \upsilon / \chi^2_\upsilon(1-\alpha/2) < \log f(\lambda) < \log f^T(\lambda) + \log \upsilon / \chi^2_\upsilon(\alpha/2)\}$$

Notes.

variance stabilized by log

set CI about mean level

simultaneous band via extreme value distribution

might take $L \to \infty$ for consistent estimate

might take weighted mean, or tapered FT

might split data, $T = LV$

More on choice of L

Consider

$$E\{f^{\mathrm{T}}(\lambda)\} = E\{\sum_l I^T(\lambda_l)/L\}, \quad \lambda_l = 2\pi r_l / T \approx \lambda = 2\pi r / T$$

$$= \int \frac{1}{L} \sum_l \frac{1}{2\pi T} \left[\sin T(\lambda_l - \alpha)/2)/(\sin(\lambda_l - \alpha)/2)\right]^2 f(\alpha) d\alpha$$

$$= \int \mathrm{W}^{\mathrm{T}}(\lambda - \alpha) f(\alpha) d\alpha$$

Choice of L affects width of $W^T(.)$

$L 2\pi / T$ radians

If f is not constant, f^T biased

Approximation to bias

Suppose $W^T(\alpha) = B_T^{-1} W(B_T^{-1}\alpha)$

$$\int W^T(\alpha) f(\lambda - \alpha) d\alpha$$

$$= \int W(\beta) f(\lambda - B_T \beta)$$

$$= \int W(\beta)[f(\lambda) - B_T \beta f'(\lambda) + B_T^2 \beta^2 f''(\lambda)/2 + \ldots] d\beta$$

$$= f(\lambda)\int W(\beta)d\beta - B_T f'(\lambda)\int \beta W(\beta)d\beta + B_T^2 f''(\lambda)\int \beta^2 W(\beta)d\beta/2 + \ldots$$

For W symmetric $\int \beta W(\beta)d\beta = 0$

Indirect estimate

$$\frac{1}{2\pi} \int \Sigma \exp\{-i\lambda u\} w^T(u) c_{xx}^T(u) du$$

Estimation of finite dimensional θ.

approximate likelihood (assuming I^T values independent exponentials)

spectrum $f(\lambda;\theta)$, θ in Θ

$$L(\theta) = \prod_r f(2\pi r/T)^{-1} \exp\{-I^T(2\pi r/T)/f(2\pi r/T)\}$$

Bivariate case.

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \int \exp\{it\lambda\} \begin{bmatrix} dZ_X(\lambda) \\ dZ_Y(\lambda) \end{bmatrix}$$

$$\mathrm{cov}\{dZ_X(\lambda), dZ_Y(\mu)\} = \delta(\lambda - \mu) f_{XY}(\lambda) d\mu$$

spectral density matrix

$$\begin{bmatrix} f_{XX}(\lambda) & f_{XY}(\lambda) \\ f_{YX}(\lambda) & f_{YY}(\lambda) \end{bmatrix}$$

Crossperiodogram.

$$I^T_{XY}(\lambda) = \frac{1}{2\pi T} d^T_X(\lambda)\overline{d^T_Y(\lambda)}$$

matrix form $\mathbf{I}^T(\lambda) = \begin{bmatrix} I^T_{XX} & I^T_{XY} \\ I^T_{YX} & I^T_{YY} \end{bmatrix}$

Smoothed periodogram.

$$\mathbf{f}^T_{NN}(\lambda) = \sum_l \mathbf{I}^T(2\pi r_l / T) / L, \quad 2\pi r_l / T \approx \lambda, l = 1,\ldots, L$$

Complex Wishart

$$\mathbf{X}_1,\ldots,\mathbf{X}_n \sim IN_r(\mathbf{0},\Sigma)$$

$$W_r^C(n,\mathbf{\Sigma}) \sim \mathbf{W} = \sum_1^n \mathbf{X}_j \overline{\mathbf{X}_j^T}$$

diagonals chi - squared

$$E\mathbf{W} = n\mathbf{\Sigma}$$

Predicting Y via X

$$\text{predicting } dZ_Y(\lambda) \text{ by } dZ_X(\lambda)$$

$$\min_A E\,|\,dZ_Y(\lambda) - A\,dZ_X(\lambda)\,|^2$$

$$A(\lambda) = f_{YX}(\lambda)\, f_{XX}(\lambda)^{-1}, \quad \text{transfer function}$$

$$|\,A(\lambda)\,|: \text{ gain} \quad \arg A(\lambda): \text{ phase}$$

$$\text{MSE}: [1 - |\,R(\lambda)\,|^2]\, f_{YY}(\lambda)$$

$$\text{coherency}: R(\lambda) = f_{YX}(\lambda) / \sqrt{f_{XX}(\lambda)\, f_{YY}(\lambda)}$$

$$a(u) = (2\pi)^{-1} \int A(\alpha) \exp\{iu\alpha\}\, d\alpha$$

$$Y(t) \approx \int \exp\{i\lambda t\} A(\lambda)\, dZ_X(\lambda)$$

Plug in estimates.

$$A^T(\lambda) = f_{YX}^T(\lambda) f_{XX}^T(\lambda)^{-1}$$

$$|A^T(\lambda)| \quad \arg A^T(\lambda)$$

$$\text{MSE}: [1-|R^T(\lambda)|^2] f_{YY}^T(\lambda)$$

$$R^T(\lambda) = f_{YX}^T(\lambda) / \sqrt{f_{XX}^T(\lambda) f_{YY}^T(\lambda)}$$

$$\text{Density of } |R^T|^2$$

$$(1-|R|^2)^L {}_2F_1(L, L; 1; |R|^2 |R^T|^2) \frac{\Gamma(L)}{\Gamma(L-1)\Gamma(1)}$$

$$\text{If } |R|^2 = 0 \text{ approx } 100\alpha\% \text{ point}$$

$$1 - (1-\alpha)^{1/(L-1)}$$

$$E|R^T|^2 \approx 1/L$$

Large sample distributions.

$$\text{var } \log|A^T| \propto [|R|^{-2} - 1]/L$$

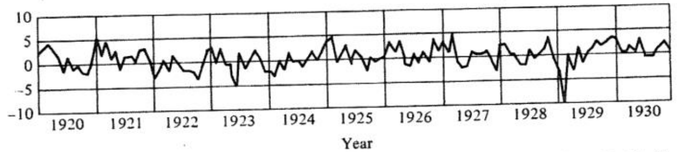
$$\text{var } \arg A^T \propto [|R|^{-2} - 1]/L$$

Berlin and Vienna monthly temperatures

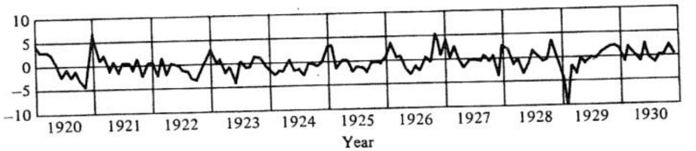
an value from the corresponding month values. If $Y(t)$ denotes the ad-
ted series for Berlin, then it is given by

$$Y(j + 12k) = B(j + 12k) - K^{-1} \sum_{k=0}^{K-1} B(j + 12k) \qquad (6.10.1)$$

0, . . . , 11; $k = 0, \ldots, K - 1$ and $K = T/12$. Let $X(t)$ likewise denote series of adjusted values for Vienna. These series are given in Figures 0.1 and 2 for 1920–1930. The original series are given in Figure 1.1.1.



ıre 6.10.1 Seasonally adjusted series of monthly mean temperatures in °C at Berlin for years 1920–1930.



ıre 6.10.2 Seasonally adjusted series of monthly mean temperatures in °C at Vienna the years 1920–1930.

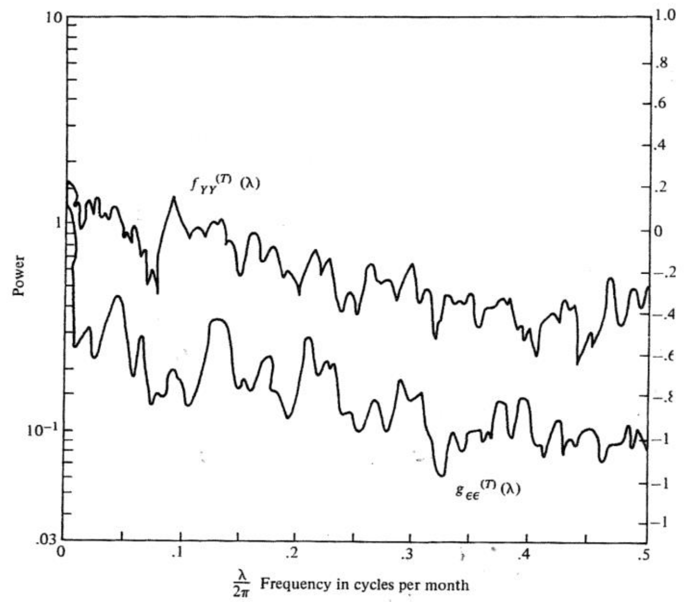


Figure 6.10.3 An estimate of the power spectrum of Berlin temperatures and an esti of the error spectrum after fitting Vienna temperatures for the years 1780–1950.

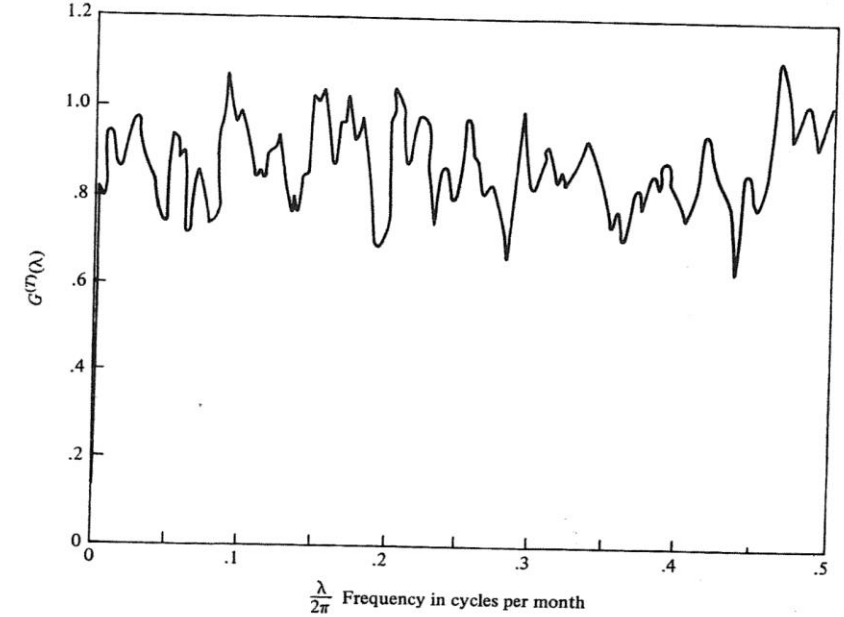


Figure 6.10.6 $G^{(T)}(\lambda)$, an estimate of the amplitude of the transfer function for fitting Berlin temperatures by Vienna temperatures.

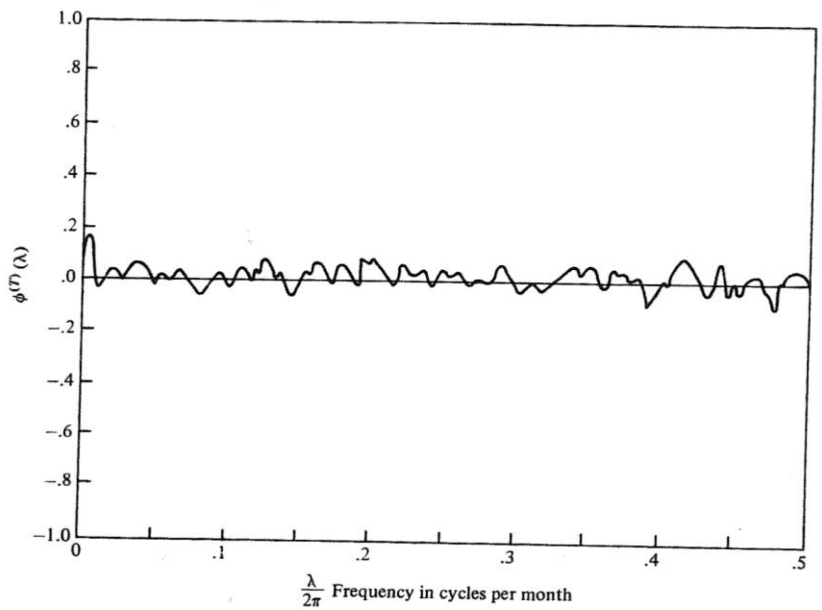


Figure 6.10.7 $\phi^{(T)}(\lambda)$, an estimate of the phase of the transfer function for fitting Berlin temperatures by Vienna temperatures.

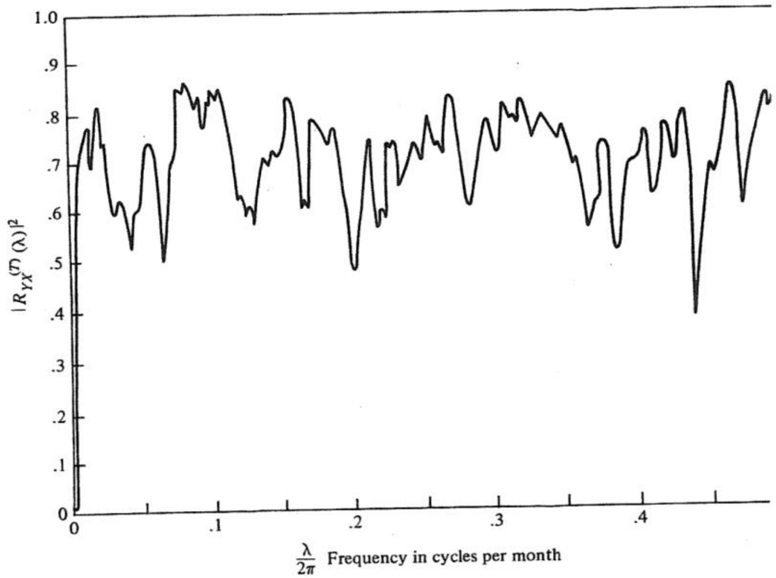


Figure 6.10.8 $|R_{YX}^{(T)}(\lambda)|^2$, an estimate of the coherence of Berlin and Vienna temperatu for the years 1780–1950.

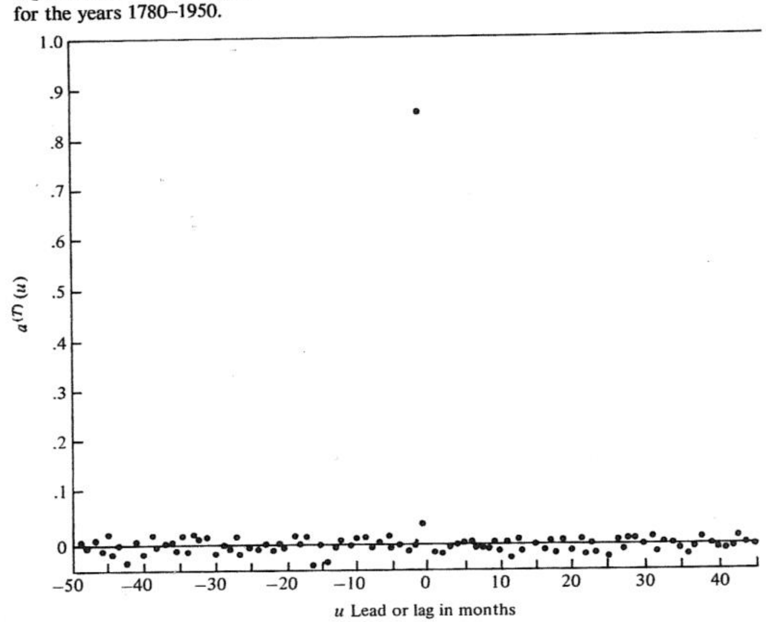
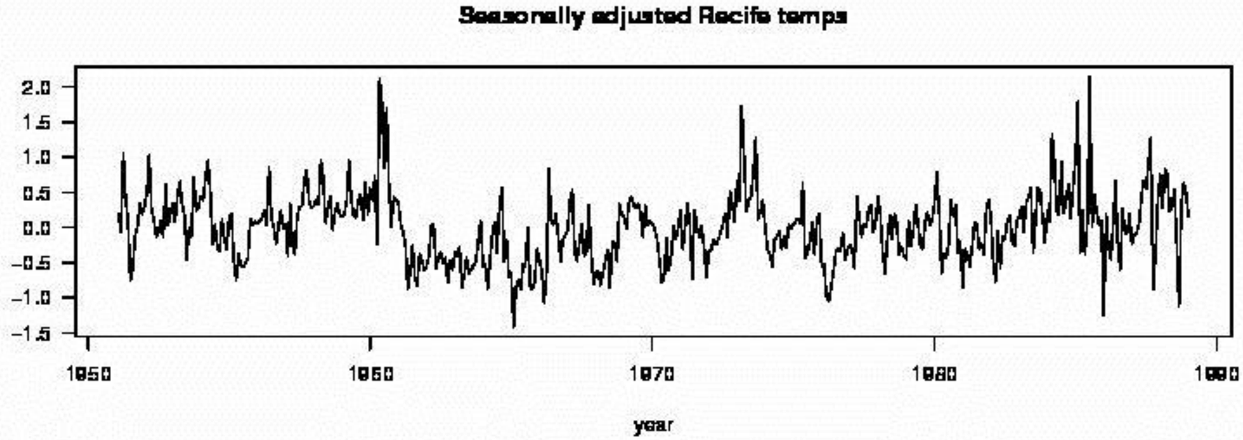
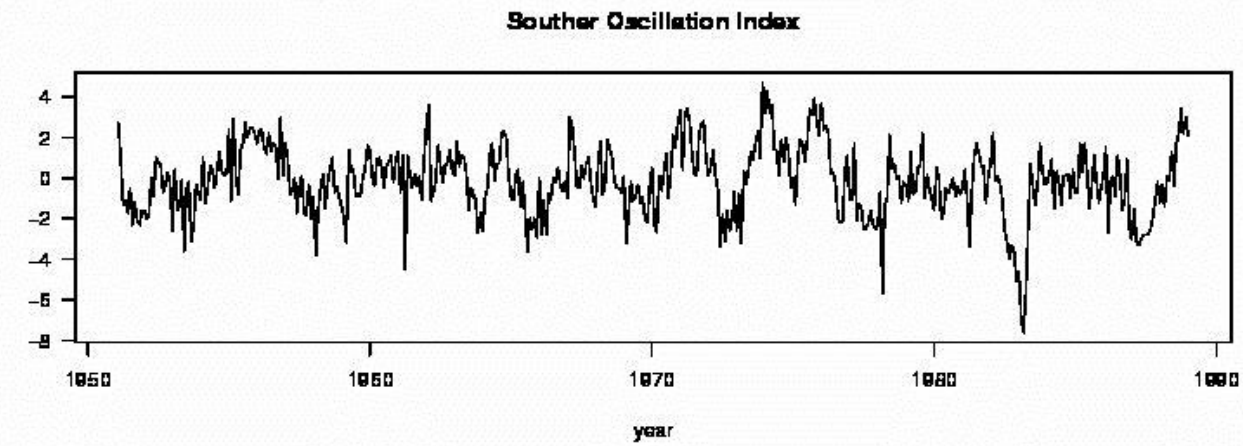


Figure 6.10.9 $a^{(T)}(u)$, an estimate of the filter coefficients for fitting Berlin temperature Vienna temperatures.

Recife
SOI





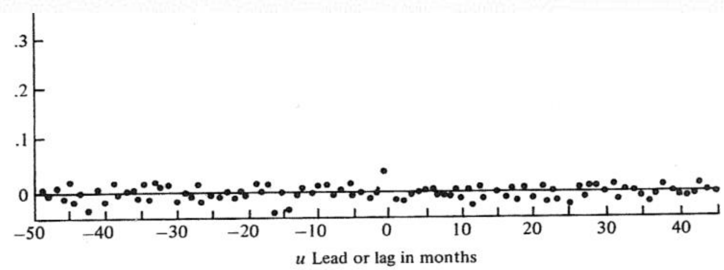
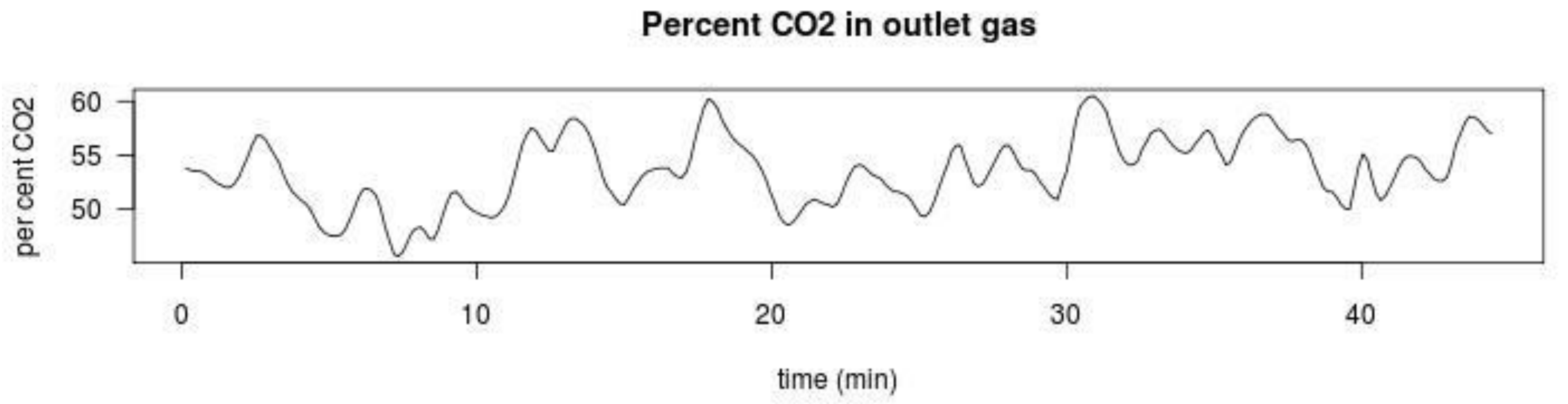
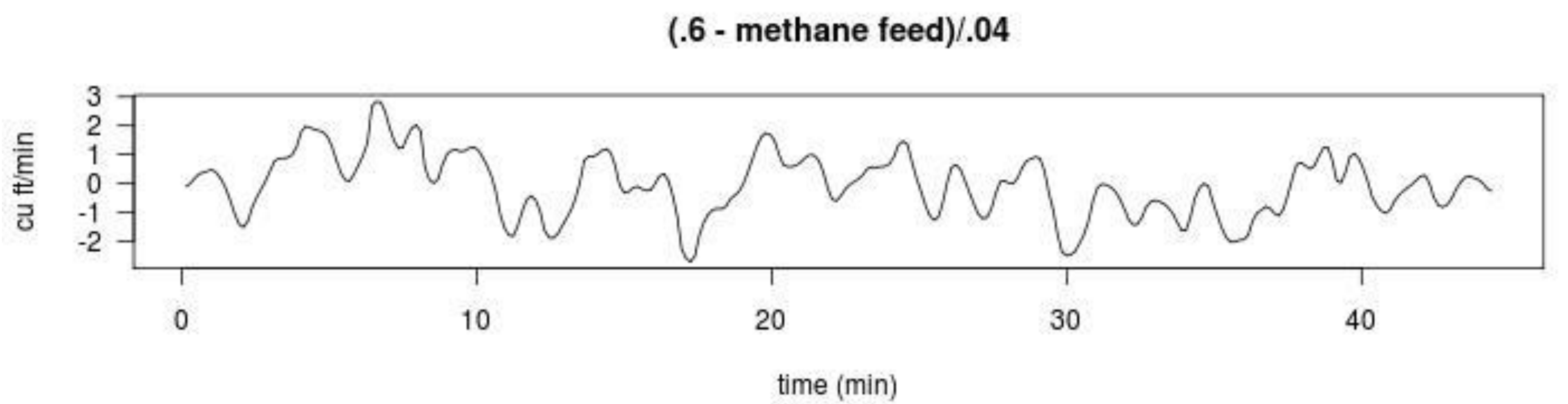
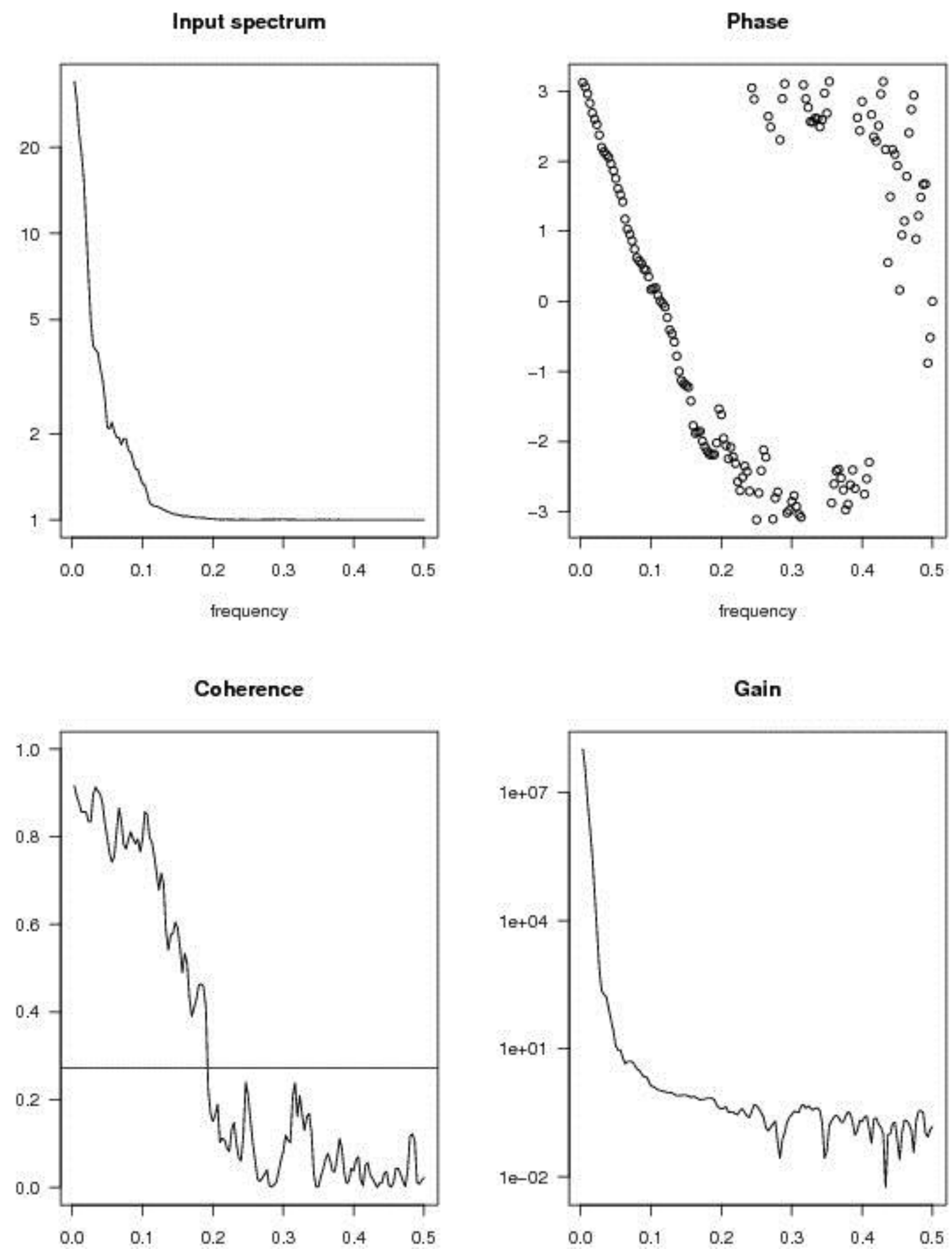


Figure 6.10.9 $a^{(T)}(u)$, an estimate of the filter coefficients for fitting Berlin temperature Vienna temperatures.

Furnace data






Input spectrum
Phase
Coherence
Gain
20
10
5
2
1
3
2
1
0
−1
−2
−3
1.0
0.8
0.6
0.4
0.2
0.0
1e+07
1e+04
1e+01
1e−02
0.0
0.1
0.2
0.3
0.4
0.5
frequency
frequency

Partial coherence/coherency. Mississipi dams

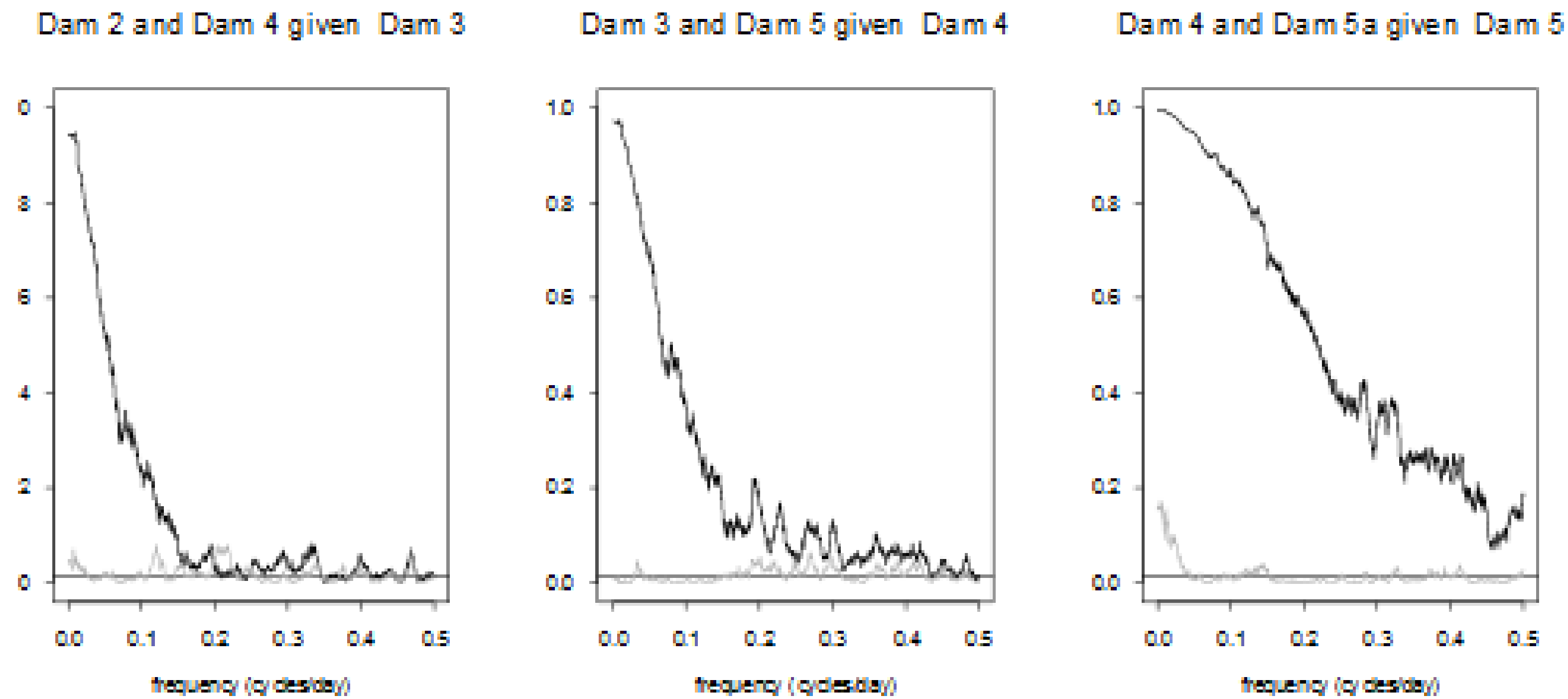
Following the connection of the dams in series one can envisage the following models for the flows at Dams 8 and 10, in terms of that at Dam 9,

$$Y_{10}(t) = \mu + \int_0^{\infty} a(t-u) Y_9(u) du + \epsilon(t)$$

$$Y_8(t) = \nu + \int^{0} b(t-u) Y_9(u) du + \eta(t)$$

$R_{XZ|Y}$ =

$(R_{XZ} – R_{XZ} \; R_{ZY})/\surd[(1- |R_{XZ}|^2)(1- |R_{ZY}|^2)]$


Dam 2 and Dam 4 given Dam 3
Dam 3 and Dam 5 given Dam 4
Dam 4 and Dam 5a given Dam 5
0.0 0.2 0.4 0.6 0.8 1.0
0.0 0.1 0.2 0.3 0.4 0.5
frequency (cycles/day)

Advantages of frequency domain approach.

techniques for many stationary processes look the same

approximate i.i.d sample values

assessing models (character of departure)

time varying variant...

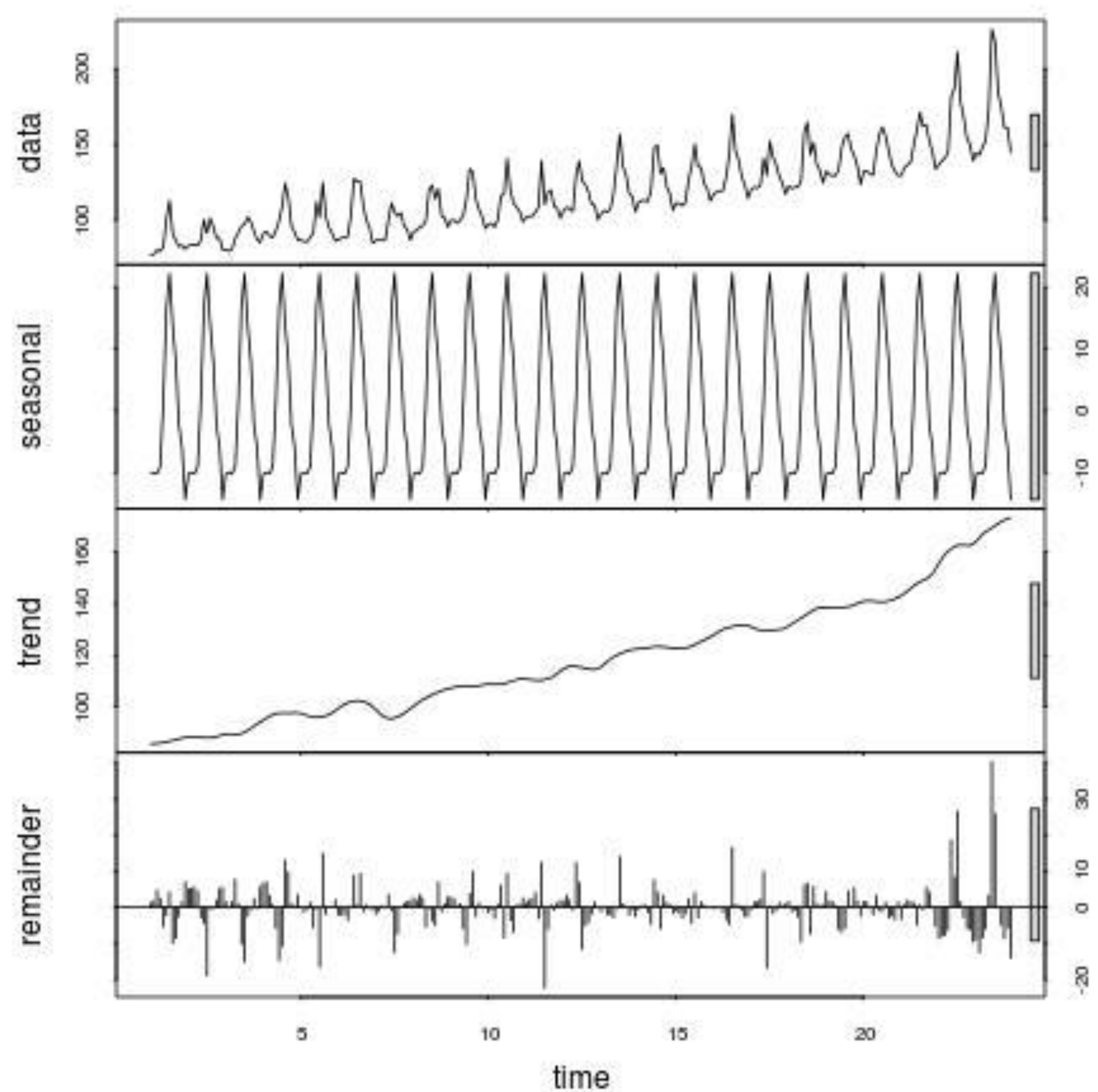
London water usage

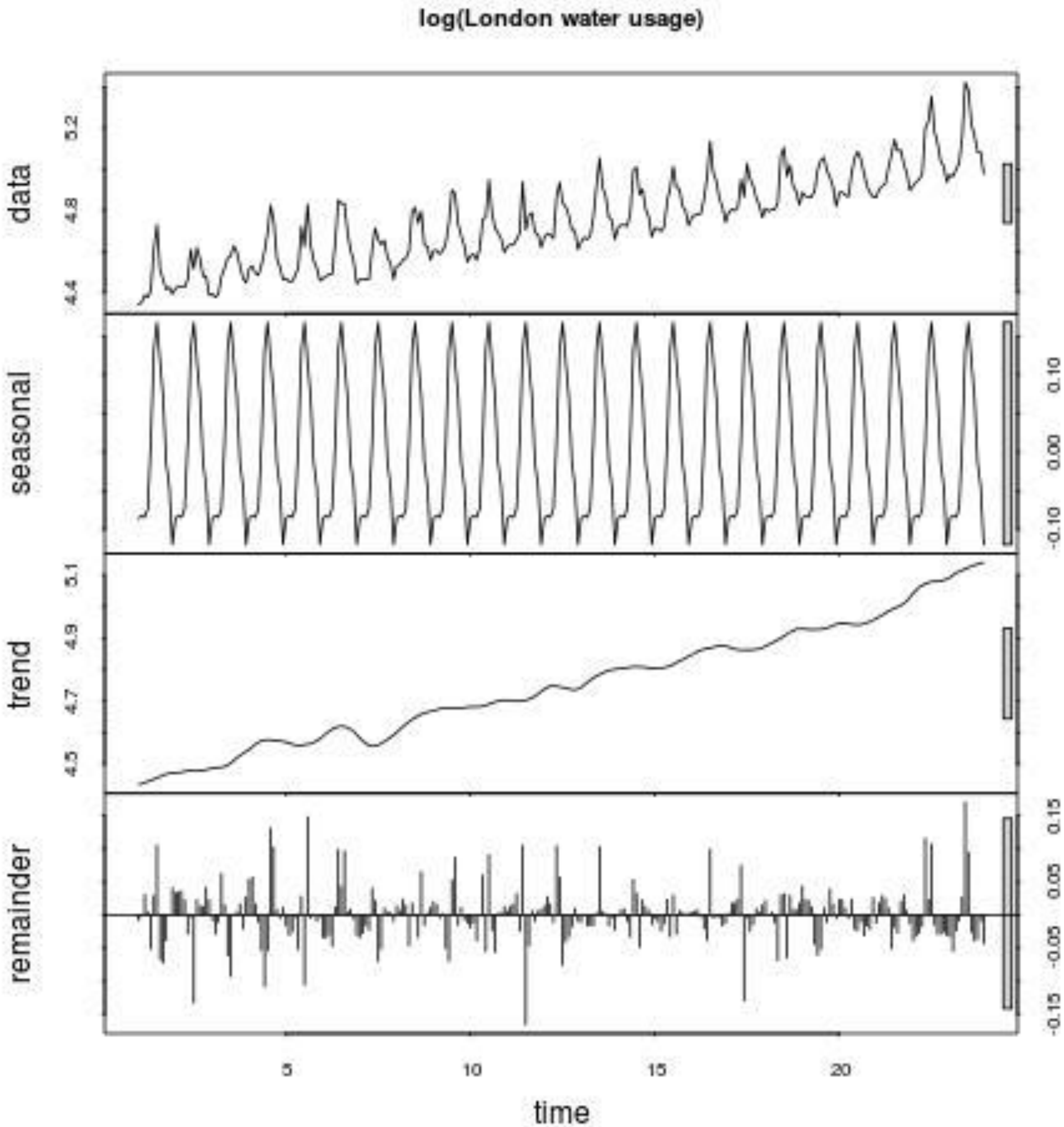
Cleveland, RB, Cleveland, WS, McRae, JE & Terpenning, I (1990), ‘STL: a seasonal-trend decomposition procedure based on loess’, Journal of Official Statistics

$$Y(t) = S(t) + T(t) + E(t)$$

Seasonal, trend. error

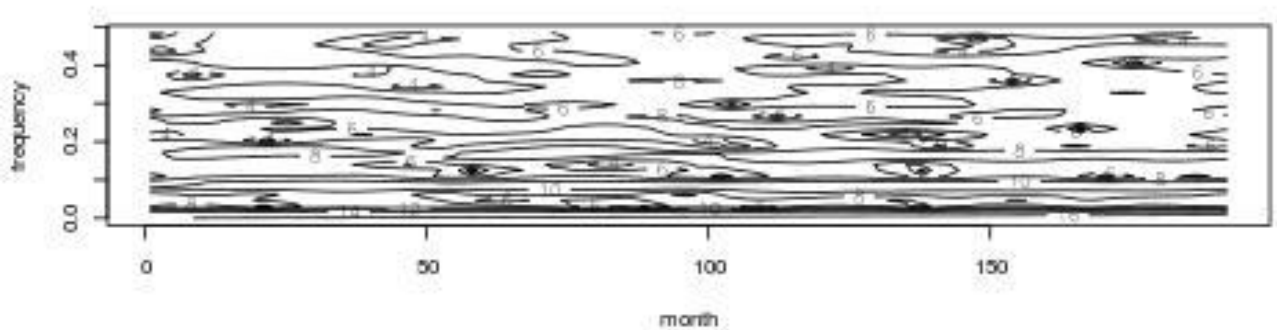
London water usage

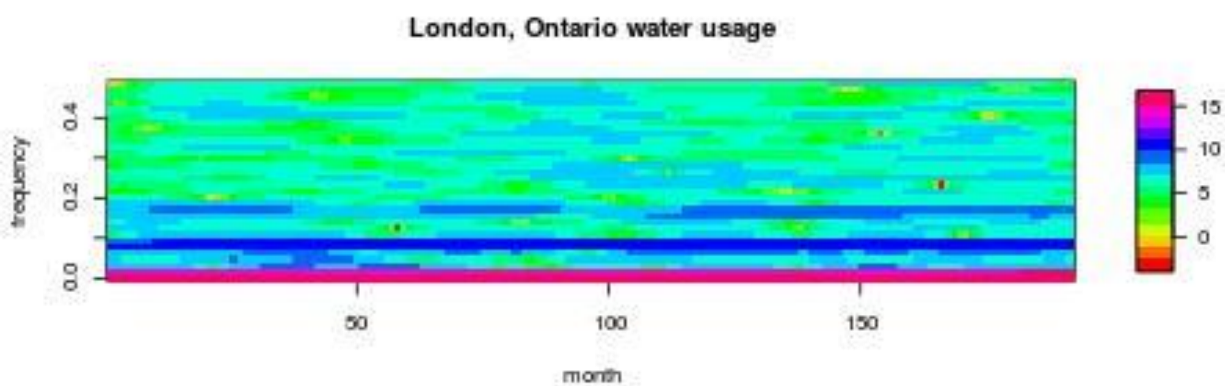




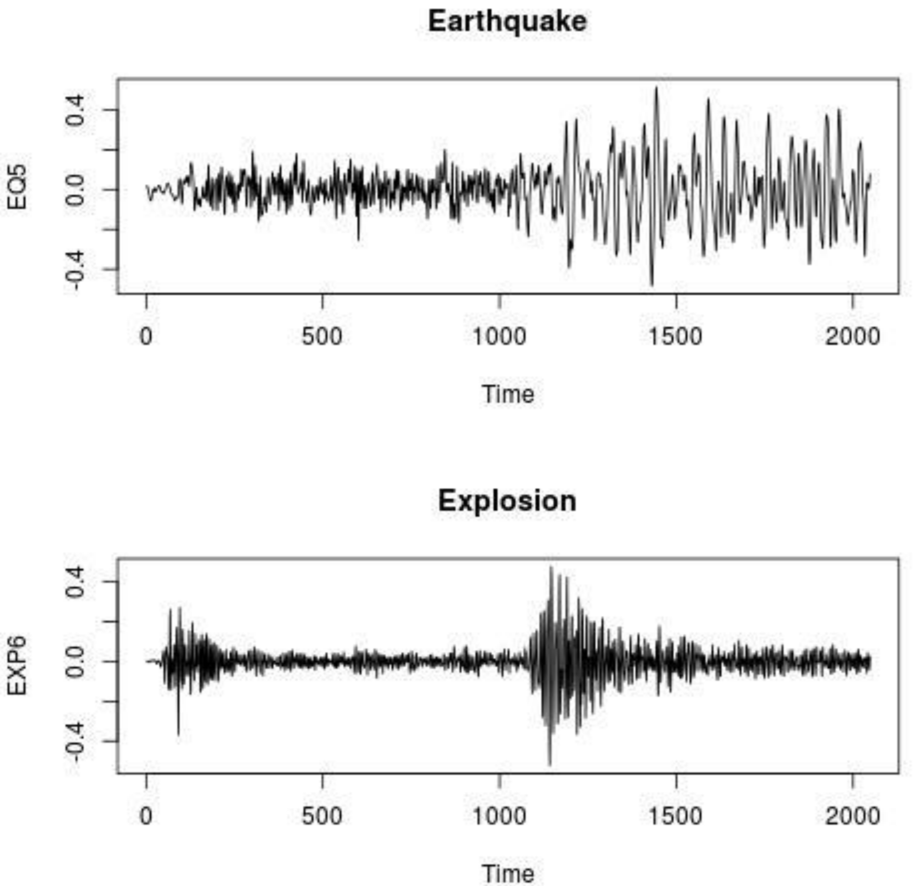
log(London water usage)
data
4.4
4.8
5.2
seasonal
-0.10
0.00
0.10
trend
4.5
4.7
4.9
5.1
remainder
-0.15
-0.05
0.05
0.15
5
10
15
20
time

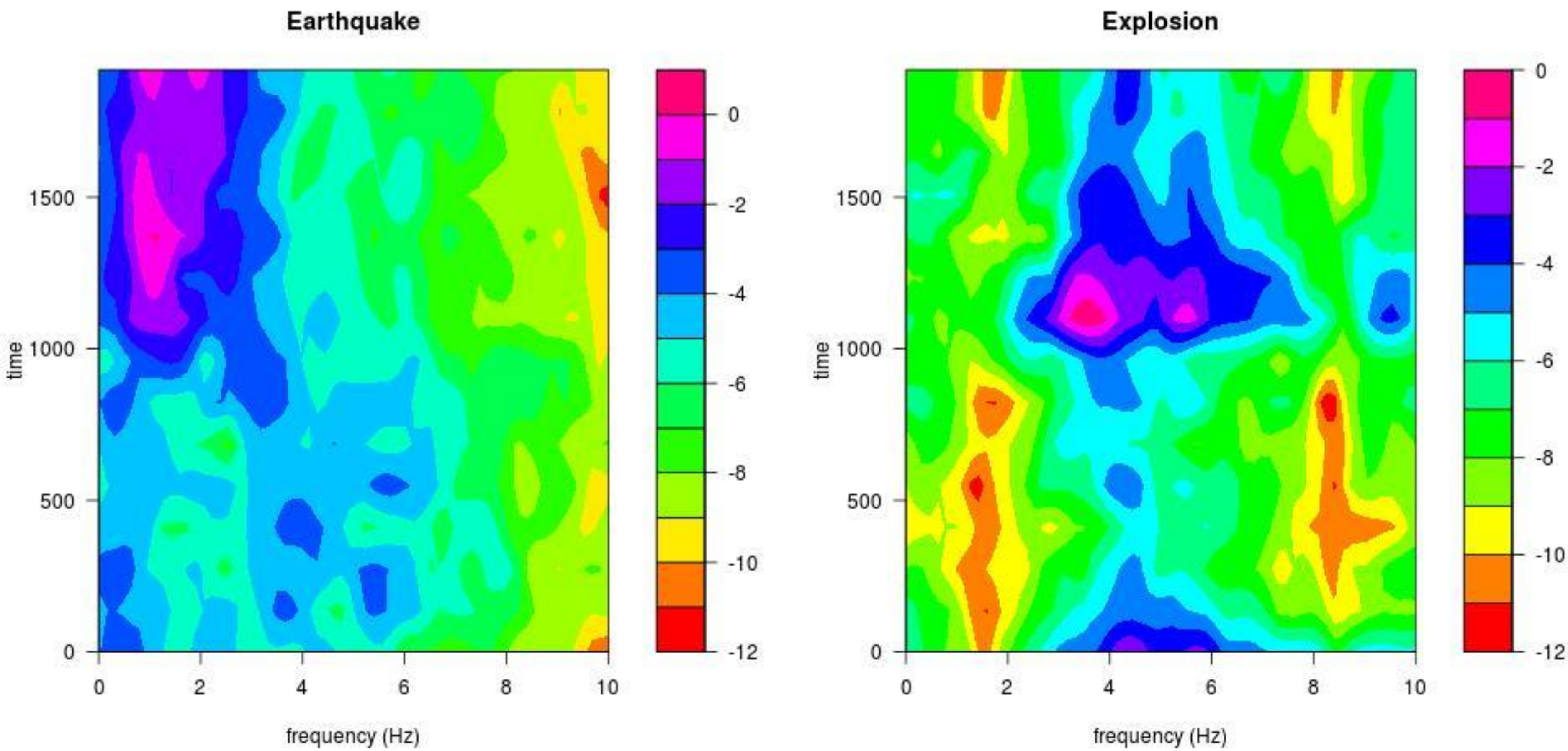
Dynamic spectrum, spectrogram: $I^T(t,\lambda)$. London water



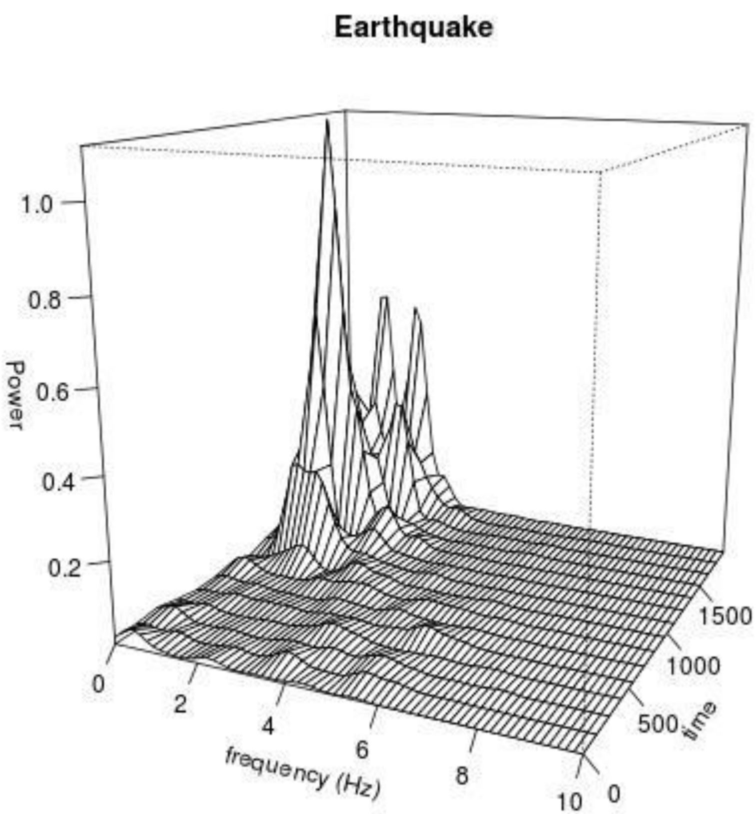


Earthquake? Explosion?

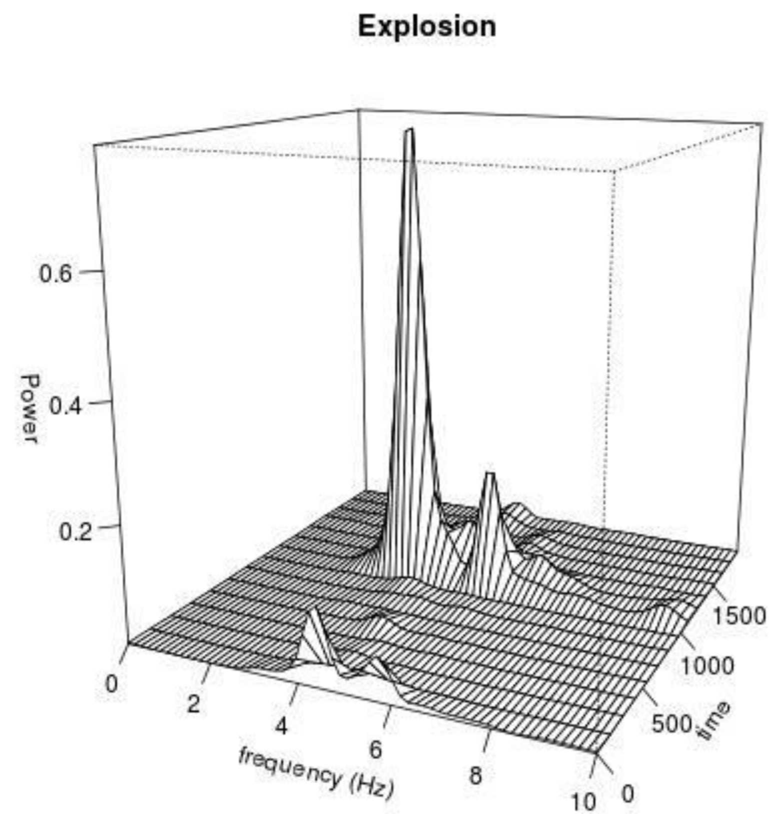




Earthquake
Explosion
time
frequency (Hz)
0
500
1000
1500
0
2
4
6
8
10
0
-2
-4
-6
-8
-10
-12

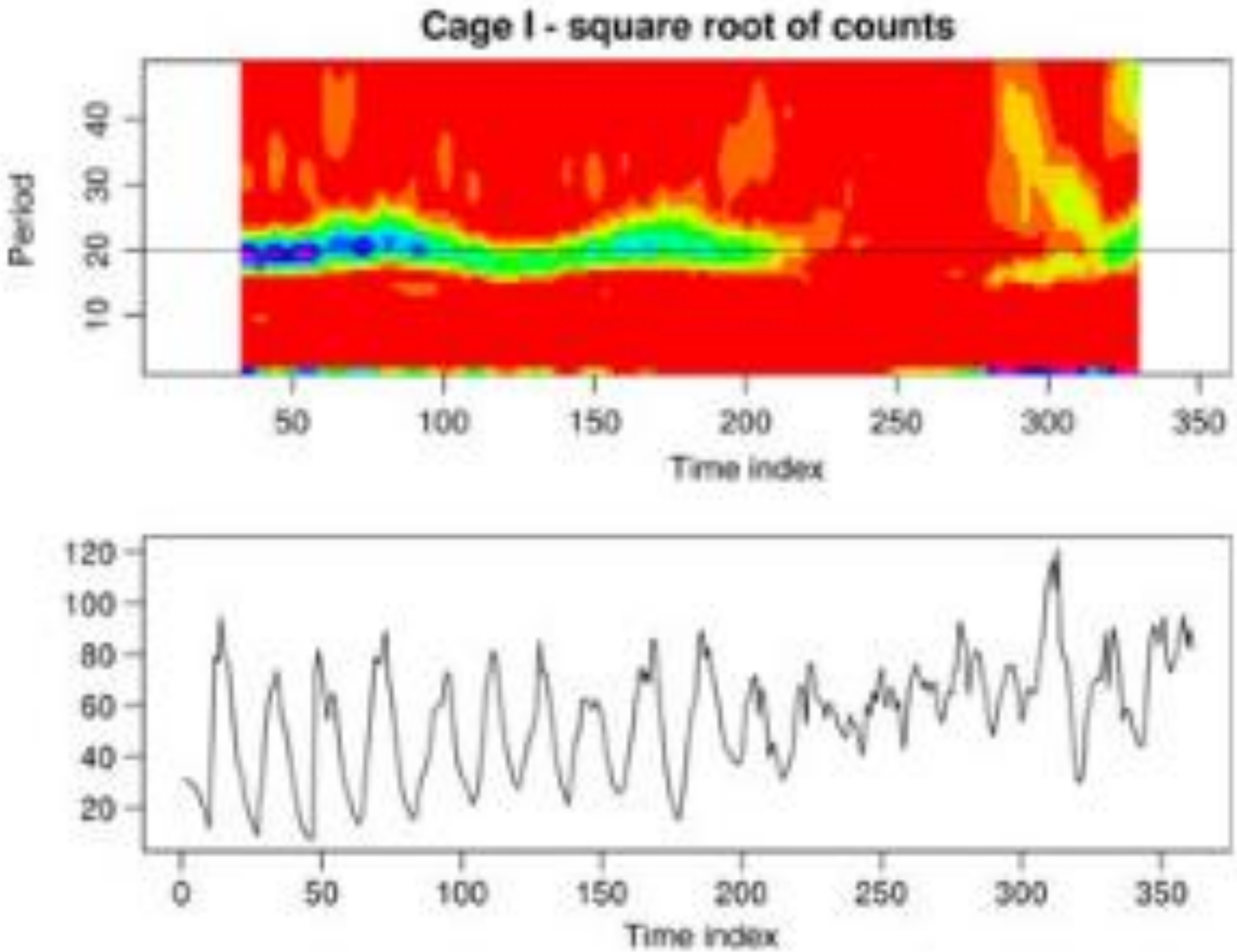


Earthquake
1.0
0.8
0.6
0.4
0.2
Power
0
2
4
6
8
10
frequency (Hz)
0
500
1000
1500
time



Explosion
0.6
0.4
0.2
Power
0
2
4
6
8
10
frequency (Hz)
0
500
1000
1500
time

Lucilia cuprina



```
nobs = length(EXP6) # number of observations
 wsize = 256 # window size
 overlap = 128 # overlap
ovr = wsize-overlap
 nseg = floor(nobs/ovr)-1; # number of segments
krnl = kernel("daniell", c(1,1)) # kernel
ex.spec = matrix(0, wsize/2, nseg)
for (k in 1:nseg)
{ a = ovr*(k-1)+1
 b = wsize+ovr*(k-1)
ex.spec[,k] = mvspec(EXP6[a:b], krnl, taper=.5, plot=FALSE)$spec }
 x = seq(0, 10, len = nrow(ex.spec)/2)
y = seq(0, ovr*nseg, len = ncol(ex.spec))
z = ex.spec[1:(nrow(ex.spec)/2),] # below is text version
filled.contour(x,y,log(z),ylab="time",xlab="frequency (Hz)",nlevels=12
,col=gray(11:0/11),main="Explosion")
dev.new() # a nicer version with color
filled.contour(x, y, log(z), ylab="time", xlab="frequency(Hz)", main=
"Explosion") dev.new() # below not shown in text
persp(x,y,z,zlab="Power",xlab="frequency(Hz)",ylab="time",ticktype="det
ailed",theta=25,d=2,main="Explosion")
```