Asymptotic normality of finite FT, Cramer and narrow band filtering

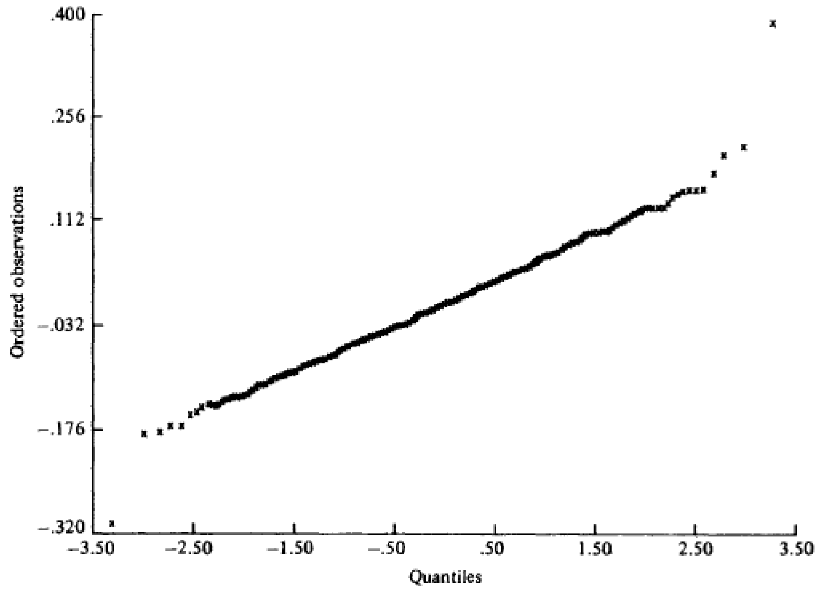


Figure 4.4.1 Normal probability plot of real part of discrete Fourier transform of seasonally adjusted Vienna mean monthly temperatures 1780–1950.

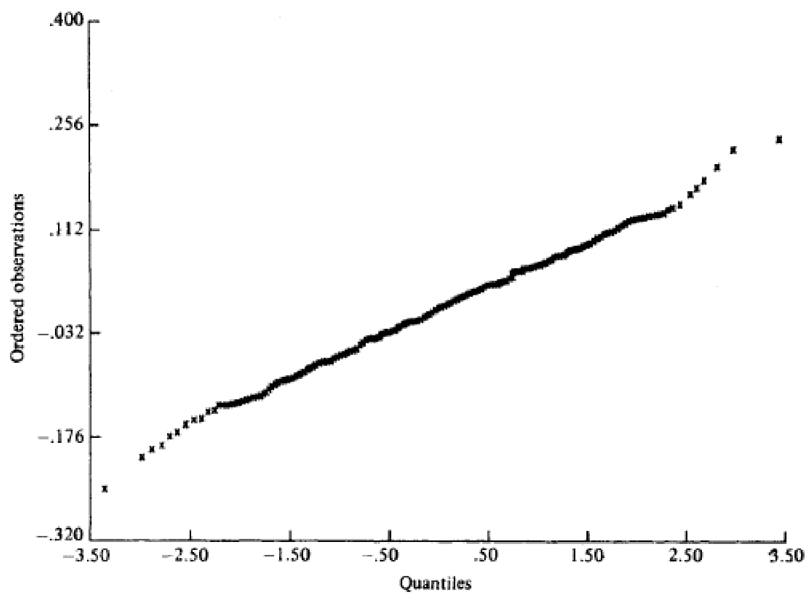


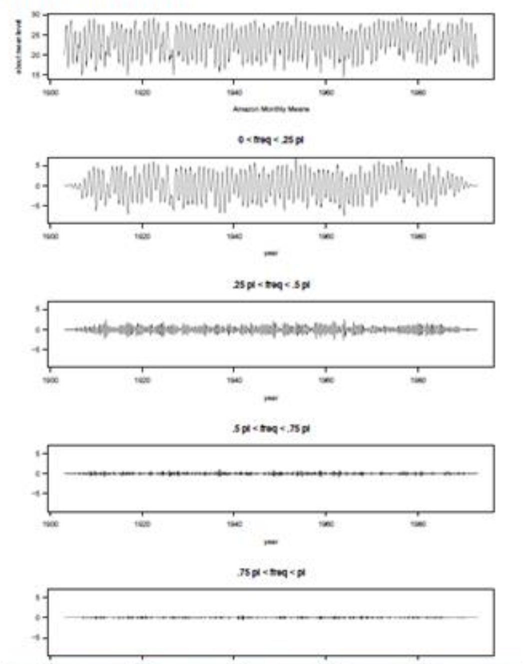
Figure 4.4.2 Normal probability plot of imaginary part of discrete Fourier transform of seasonally adjusted Vienna mean monthly temperatures 1780–1950.

Narrow bandpass filtering

$$T^{-1}\, d_X^{(T)}\left(\frac{2\pi s}{T}\right) \exp\left\{i\frac{2\pi st}{T}\right\} + T^{-1}\, d_X^{(T)}\left(-\frac{2\pi s}{T}\right) \exp\left\{-i\frac{2\pi st}{T}\right\}$$

$$t = 0, \pm 1, \ldots .$$

Amazon River at Manaus



Cramer representation

$$\mathbf{d}_X^{(T)}(\lambda) = \sum_{t=-T}^{T} \mathbf{X}(t) \exp\{-i\lambda t\}. \qquad (4.6.2)$$

This transform will provide the basis for the representation. Set

$$2\pi\ \mathbf{Z}_X^{(T)}(\lambda) = \int_0^{\lambda} \mathbf{d}_X^{(T)}(\alpha)d\alpha. \qquad (4.6.3)$$

We see

$$2\pi\ \mathbf{Z}_X^{(T)}(\lambda) = \sum_{t=-T}^{T} \mathbf{X}(t)[1 - \exp\{-i\lambda t\}]/(-it) \qquad (4.6.4)$$

if we understand

$$[1 - \exp\{-i\lambda t\}]/(-it) = \lambda \qquad \text{for } t = 0. \qquad (4.6.5)$$

Define

$$\eta(\lambda) = \sum_{j=-\infty}^{\infty} \delta(\lambda + 2\pi j) \qquad (4.6.6)$$

to be the period 2π extension of the Dirac delta function. We may now state

Theorem 4.6.1 Let $\mathbf{X}(t)$, $t = 0, \pm 1, \ldots$ satisfy Assumption 2.6.1. Let $\mathbf{Z}_X^{(T)}(\lambda)$, $-\infty < \lambda < \infty$, be given by (4.6.4). Then there exists $\mathbf{Z}_X(\lambda)$, $-\infty < \lambda < \infty$, such that $\mathbf{Z}_X^{(T)}(\lambda)$ tends to $\mathbf{Z}_X(\lambda)$ in mean of order ν, for any $\nu > 0$. Also $\mathbf{Z}_X(\lambda + 2\pi) = \mathbf{Z}_X(\lambda)$, $\overline{\mathbf{Z}_X(\lambda)} = \mathbf{Z}_X(-\lambda)$ and

$$\text{cum}\{Z_{a_1}(\lambda_1), \ldots, Z_{a_k}(\lambda_k)\} = \int_0^{\lambda_1} \cdots \int_0^{\lambda_k} \eta\left(\sum_1^k \alpha_j\right) f_{a_1 \ldots a_k}(\alpha_1, \ldots, \alpha_{k-1}) d\alpha_1 \cdots d\alpha_k \qquad (4.6.7)$$

for $a_1, \ldots, a_k = 1, \ldots, r$; $k = 2, 3, \ldots$.

$$\text{cov}\{d\mathbf{Z}_X(\lambda), d\mathbf{Z}_X(\mu)\} = \eta(\lambda - \mu)\, \mathbf{f}_{XX}(\lambda) d\lambda d\mu \qquad (4.6.9)$$

$$\int_0^{2\pi} \phi(\lambda) d\mathbf{Z}_X(\lambda). \tag{4.6.10}$$

If

$$\int_0^{2\pi} \phi(\lambda)\mathbf{f}_{XX}(\lambda)\overline{\phi(\lambda)}^\tau d\lambda < \infty, \tag{4.6.11}$$

this integral exists when defined as

$$\underset{N\to\infty}{\text{l.i.m}} \frac{2\pi}{N} \sum_{n=0}^{N-1} \phi\left(\frac{2\pi n}{N}\right)\left[\mathbf{Z}_X\left(\frac{2\pi(n+1)}{N}\right) - \mathbf{Z}_X\left(\frac{2\pi n}{N}\right)\right]. \tag{4.6.12}$$

See Cramér and Leadbetter (1967) Section 5.3. We may now state the **Cramér representation** of the series $\mathbf{X}(t)$, $t = 0, \pm 1, \ldots$.

Theorem 4.6.2 Under the conditions of Theorem 4.6.1

$$\mathbf{X}(t) = \int_0^{2\pi} \exp\{i\lambda t\} d\mathbf{Z}_X(\lambda) \qquad t = 0, \pm 1, \ldots \tag{4.6.13}$$

with probability 1, where $\mathbf{Z}_X(\lambda)$ satisfies the properties indicated in Theorem 4.6.1.

$$\mathbf{Y}(t) = \sum_u \mathbf{a}(t-u)\mathbf{X}(u) \qquad t = 0, \pm 1, \ldots$$

where the series $\mathbf{X}(t)$ has Cramér representation (4.6.13). If

$$\mathbf{A}(\lambda) = \sum_u \mathbf{a}(u) \exp\{-i\lambda u\} \qquad -\infty < \lambda < \infty,$$

with

$$\int_0^{2\pi} \mathbf{A}(\lambda)\mathbf{f}_{XX}(\lambda)\overline{\mathbf{A}(\lambda)}^\tau d\lambda < \infty,$$

then

$$\mathbf{Y}(t) = \int_0^{2\pi} \exp\{i\lambda t\}\mathbf{A}(\lambda) d\mathbf{Z}_X(\lambda) \qquad t = 0, \pm 1, \ldots$$

In differential notation the latter may be written

$$d\mathbf{Z}_Y(\lambda) = \mathbf{A}(\lambda) d\mathbf{Z}_X(\lambda) \qquad -\infty < \lambda < \infty.$$

$$\mathbf{f}_{YY}(\lambda) = \mathbf{A}(\lambda)\mathbf{f}_{XX}(\lambda)\overline{\mathbf{A}(\lambda)}^{\tau}$$

of Section 2.8.

Suppose the filter is a band-pass filter with transfer function,

$$\begin{aligned} A(\lambda) &= 1 \quad \text{for } |\lambda \pm \omega| < \Delta \\ &= 0 \quad \text{otherwise} \end{aligned}$$

applied to each coordinate of the series $\mathbf{X}(t)$, $t = 0, \pm 1, \ldots$.

Suppose, as we may, that the Cramér representation of $\mathbf{X}$

$$\mathbf{X}(t) = \int_{-\pi}^{\pi} \exp\{i\lambda t\} d\mathbf{Z}_X(\lambda).$$

Then the band-pass filtered series may be written

$$\begin{aligned} \mathbf{Y}(t) &= \left\{ \int_{-\omega-\Delta}^{-\omega+\Delta} + \int_{\omega-\Delta}^{\omega+\Delta} \right\} \exp\{i\lambda t\} d\mathbf{Z}_X(\lambda) \\ &\doteq \exp\{i\omega t\} d\mathbf{Z}_X(\omega) + \exp\{-i\omega t\} d\mathbf{Z}_X(-\omega) \end{aligned}$$

$$\mathbf{d}_X{}^{(T)}(\lambda) = \sum_t h\left(\frac{t}{T}\right)\mathbf{X}(t) \exp\{-i\lambda t\}$$

for some tapering function $h(u)$. By direct substitution we see that

$$\mathbf{d}_X{}^{(T)}(\lambda) = \int_0^{2\pi} H^{(T)}(\lambda - \alpha) d\mathbf{Z}_X(\alpha)$$

$$H^{(T)}(\lambda) = \sum_t h\left(\frac{t}{T}\right) \exp\{-i\lambda t\}.$$