Power spectral density. frequency-side, λ, vs. time-side, t

$\lambda/2\pi$: *frequency* (cycles/unit time)

$$f_{NN}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-i\lambda u\}[\delta(u)\, p_N + q_{NN}(u)]du$$

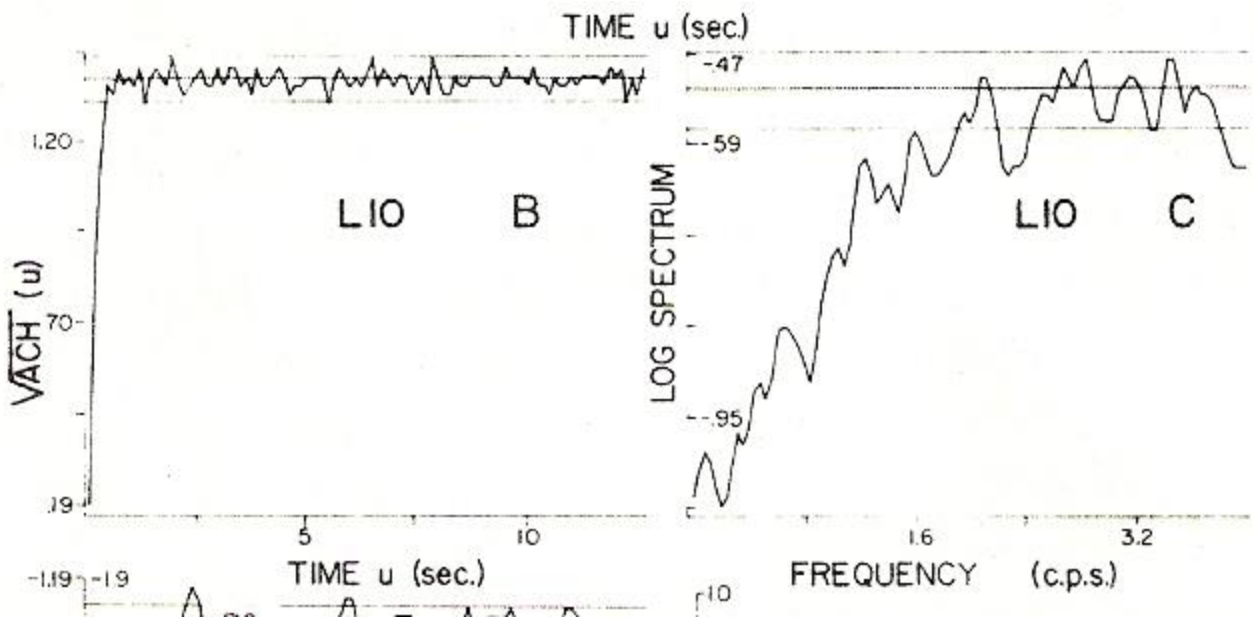
$$= \frac{1}{2\pi}\, p_N + \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\{-i\lambda u\} q_{NN}(u)du$$

$$\sim \frac{1}{2\pi}\, p_N \text{ for large } |\lambda|$$

Non-negative

Unifies analyses of processes of widely varying types

Examples.



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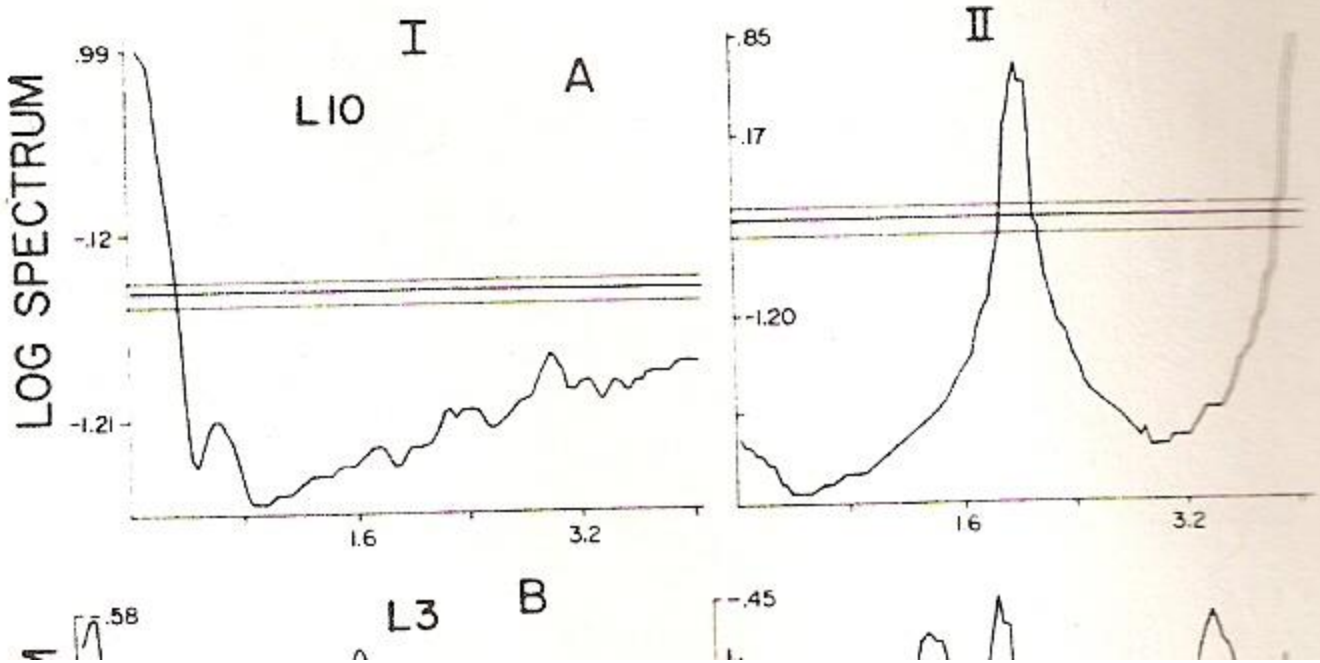
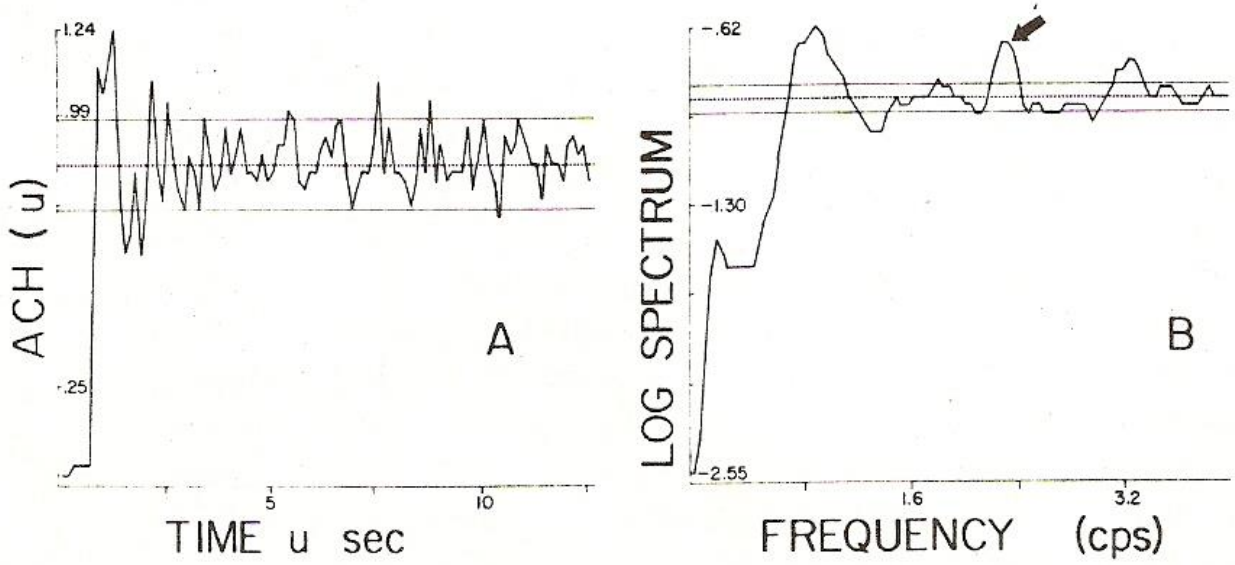

I
A
L10
LOG SPECTRUM
.99
-.12
-1.21
1.6
3.2
II
.85
.17
-1.20
1.6
3.2
B
L3
.58
.45


1.24
.99
.25
ACH (u)
A
5
10
TIME u sec
-.62
-1.30
-2.55
LOG SPECTRUM
B
1.6
3.2
FREQUENCY (cps)

Spectral representation. stationary increments - Kolmogorov

$$N(t) = \int_{-\infty}^{\infty} \frac{\exp\{it\lambda\} - 1}{i\lambda} dZ_N(\lambda)$$

$$dN(t)/dt = \int_{-\infty}^{\infty} \exp\{it\lambda\} dZ_N(\lambda)$$

Z_N : random, complex - valued, $\overline{\mathrm{dZ_N}(\boldsymbol{\lambda})} = dZ_N(-\boldsymbol{\lambda})$

$$E\{dZ_N(\boldsymbol{\lambda})\} = \boldsymbol{\delta}(\boldsymbol{\lambda}) d\boldsymbol{\lambda}$$

$$cum\{dZ_N(\boldsymbol{\lambda}_1),\ldots,dZ_N(\boldsymbol{\lambda}_K)\} = \boldsymbol{\delta}(\boldsymbol{\lambda}_1 + \ldots + \boldsymbol{\lambda}_K) f_{N\ldots N}(\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_{K-1}) d\boldsymbol{\lambda}_1 \ldots d\boldsymbol{\lambda}_K$$

cumulant spectrum of order K

$$\mathrm{cov}\{dZ_N(\boldsymbol{\lambda}), dZ_N(\boldsymbol{\mu})\} = \boldsymbol{\delta}(\boldsymbol{\lambda} - \boldsymbol{\mu}) f_{NN}(\boldsymbol{\lambda}) d\boldsymbol{\lambda} d\boldsymbol{\mu}$$

orthogonal increments

$$\mathrm{cov}\{X, Y\} = E\{(X - \boldsymbol{\mu}_X)\overline{(Y - \boldsymbol{\mu}_Y)}\}$$

Frequency domain approach. Coherency, coherence

Cross-spectrum.

$$f_{MN}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-i\lambda u\} q_{MN}(u)\,du$$

Coherency.

$$R_{MN}(\lambda) = f_{MN}(\lambda)/\sqrt{\{f_{MM}(\lambda)\, f_{NN}(\lambda)\}}$$

complex-valued, 0 if denominator 0

Coherence

$$|R_{MN}(\lambda)|^2 = |f_{MN}(\lambda)|^2 / \{f_{MM}(\lambda)\, f_{NN}(\lambda)|$$

$|R_{MN}(\lambda)|^2 \leq 1$, c.p. multiple R^2

Proof. *Filtering*. $M = \{\sigma_j\}$

$$\int a(t-v)dM(v) = \Sigma\, a(t-\sigma_j)$$

Consider

$$dO(t) = dN(t) - \int a(t-v)dM(v)dt, \quad \text{(stationary increments)}$$

$$f_{OO} = f_{NN} - Af_{MN} - \overline{f_{NM}\,A} + Af_{MM}\,\overline{A}$$

where

$$A(\lambda) = \int \exp\{-i\lambda u\}a(u)du$$

$f_{OO}(\lambda)$ is a minimum at $A(\lambda) = f_{NM}(\lambda)f_{MM}(\lambda)^{-1}$

Minimum: $(1 - |R_{MN}(\lambda)|^2)f_{NN}(\lambda)$

$$0 \le |R_{MN}(\lambda)|^2 \le 1$$

Proof.

$$f_{OO} = f_{NN} - Af_{MN} - \overline{f_{NM}\,A} + Af_{MM}\,\overline{A}$$

Consider first and second derivatives

$$f_{OO} \geq 0$$

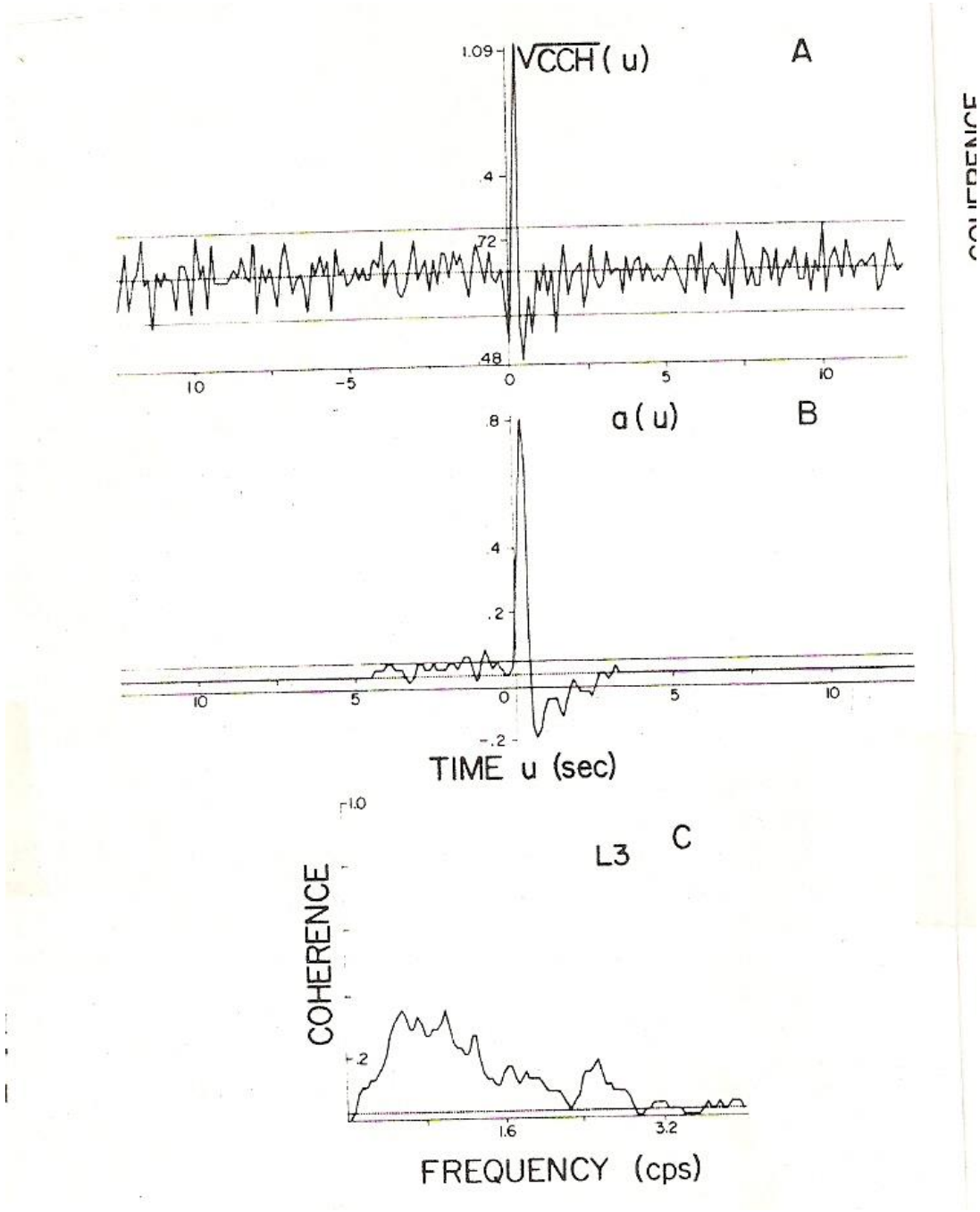
Take $A = f_{NM}\, f_{MM}^{-1}$

$$f_{NN} - f_{NM}\, f_{MM}^{-1}\, f_{MN} \geq 0$$

Coherence, measure of the linear time invariant association of the components of a stationary bivariate process.

Empirical examples.

sea hare



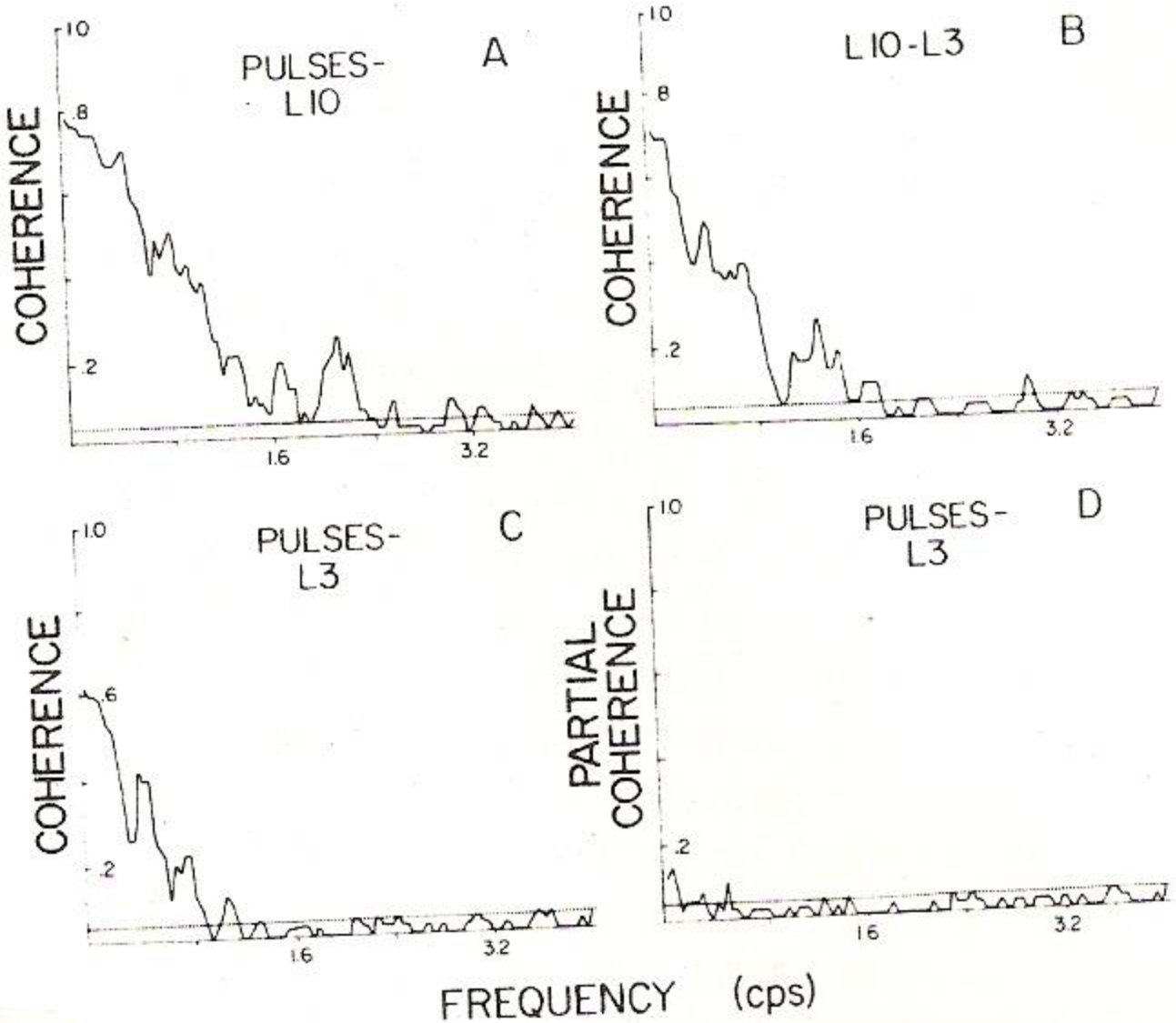
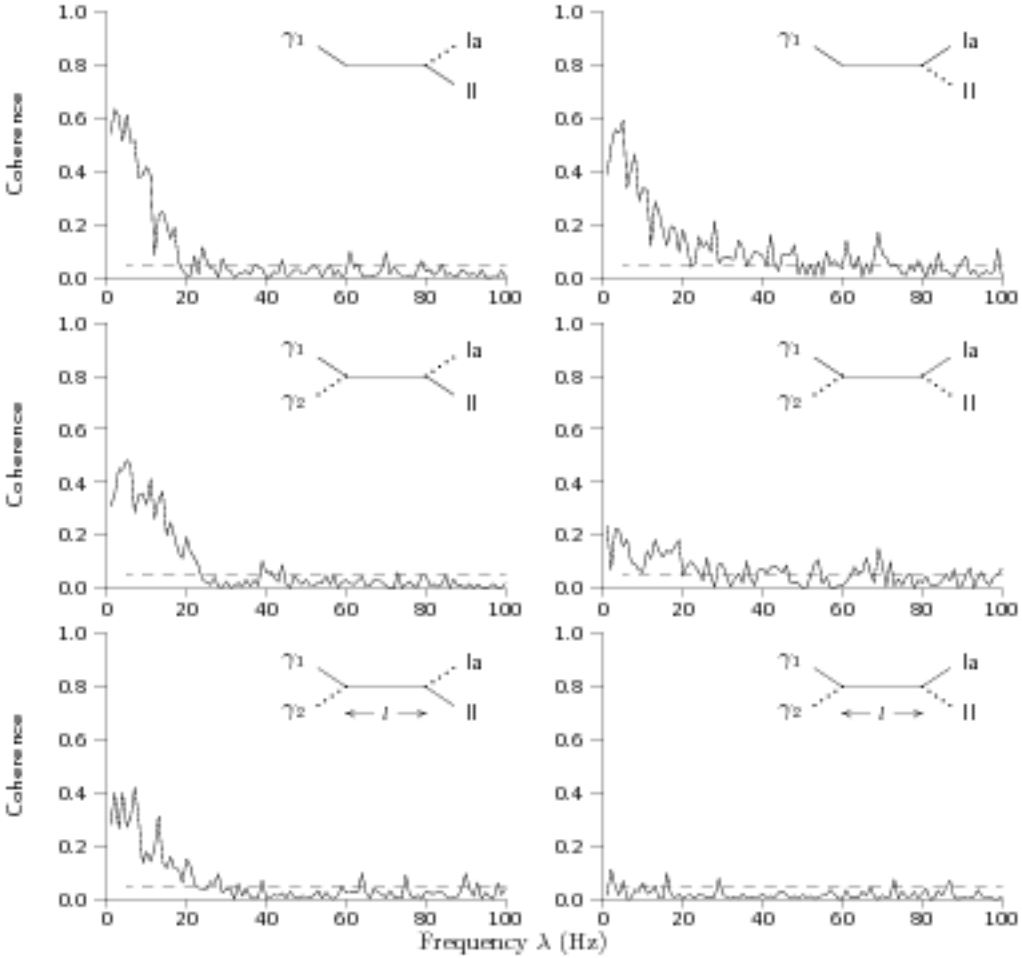


Fig. 5. Value of partial coherences. Confirmation of the a priori

Muscle spindle



Spectral representation approach.

$$dM(t)/dt = \int \exp\{it\lambda\} dZ_M(\lambda)$$

$$dN(t)/dt = \int \exp\{it\lambda\} dZ_N(\lambda)$$

$$\mathrm{cov}\{dZ_M(\lambda), dZ_N(\mu)\} = \delta(\lambda - \mu) dF_{MN}(\lambda) d\mu, \quad F_{NM} \text{ of b.v.}$$

Filtering.

$$dO(t)/dt = \int a(t-v)dM(v) = \Sigma\, a(t-\sigma_j)$$

$$= \int \exp\{it\lambda\} dZ_M(\lambda)$$

Partial coherency. Trivariate process {M,N,O}

$$f_{MN|O} = [\, f_{MN} - f_{MO} f_{ON} \,] / \sqrt{\{[1 - |\, f_{MO} \,|^2][1 - |\, f_{ON} \,|^2]\}}$$

“Removes” the linear time invariant effects of O from M and N

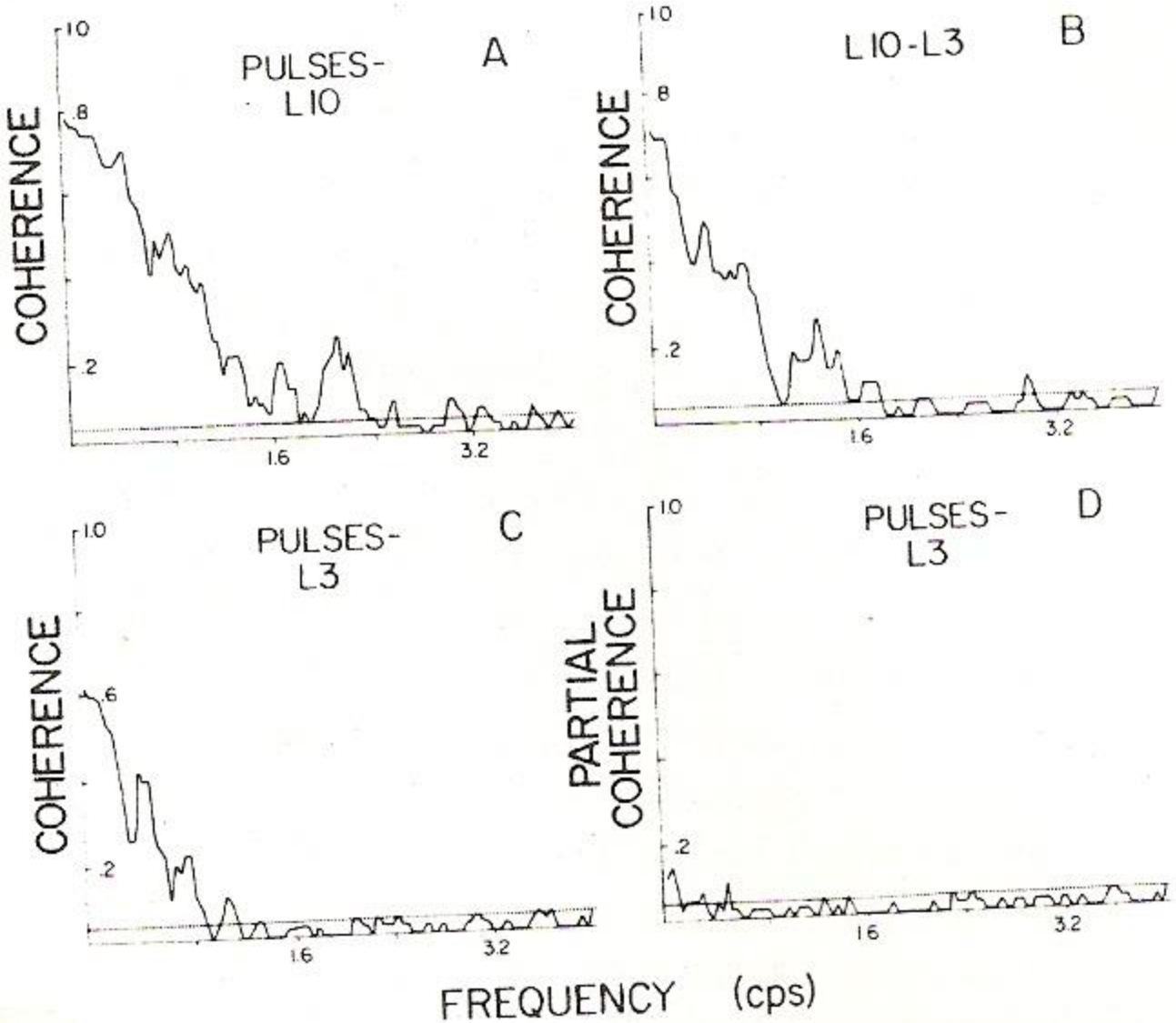


Fig. 5. Value of partial coherences. Confirmation of the a priori