Operations/systems.

Processes so far considered:

$Y(t),\ 0<t<T$	*time series*
using Dirac deltas includes	*point process*
$Y(x,y),\ 0<x<X,\ 0<y<Y$	*image*
includes	*spatial point process*
$Y(x,y,t),\ 0<x<X,\ 0<y<Y,\ 0<t<T$	*spatial-temporal*
includes	*trajectory*

and more

Manipulating process data

All functions, so can expect things computed and parameters to be functions also

Operations that are common in nature

Differential equations - Newton's Laws

Systems - input and (unique) output

box and arrow diagrams

Operations. carry one function/process into another

Domain D = {X(t), t in R^p or Z^p}

Map A[.] notice [ ]

Range {Y(.) = A[X](.), X in D}

X, Y may be vector-valued

Examples. *running mean* {X(t-1)+X(t)+X(t+1)}/3

running median Y(x,y) = median{X(u,v), u= x,x±1, v=y±1}

gradient $Y(x,y) = \nabla X(x,y)$

level crossings $Y(t) = |X'(t)|\delta(X(t)-a)$ t.s. ==> p.p.

Time invariance.

t in Z or R

translation operator

$T^u X(.) = X(.+u)$

$T^u X(t) = X(t+u)$, for all t

operator A is *time invariant* if

$A[T^u X](t) = A[X](t+u)$, for all t,u

Examples. previous slide

Non example. $Y(t) = \sup_{0<s<t} |X(s)|$

Operator A[.] is *linear* if

$$A[\alpha_1 X_1 + \alpha_2 X_2] = \alpha_1 A[X_1] + \alpha_2 A[X_2]$$

α's in R, X's in D

α, X can be complex-valued

complex numbers: $z=u + iv$, $i = \text{sqrt}(-1)$

$|z| = \text{sqrt}(u^2 + v^2)$, $\arg z=\tan^{-1}(\text{Re}(z),\text{Im}(z))$

De Moivre: $\exp\{ix\} = \cos x + i \sin x$

Linear time invariant A[.].   filter

Lemma. If {e(t)=exp{iλt}, t in Z} in D_A, then there is A(λ) with

$$A[e](t) = \exp\{i\lambda t\}A(\lambda)$$

Proof next slide

A(.): *transfer function*   notice ( )

arg(A(.)): *phase* |A(.)|: *amplitude*

Example: If $Y(t) = A[X](t) = \Sigma_u\, a(u)X(t-u)$,   u in Z

$A(\lambda) = \Sigma_u\, a(u) \exp\{-i\lambda t\}$ FT

u: *lag*   A[X]: *convolution*

Proof.

$e_\lambda(t) = \exp(i\lambda t)$

$A[e](t+u) = A[T\ e](t)$	definition
$= A[\exp\{i\lambda u\}e](t)$	definition
$= \exp\{i\lambda u\}A[e](t)$	linear
$A[e](u) = \exp\{i\lambda u\}A[e](0)$	set t = 0

$A(\lambda) = A[e_\lambda](0)$

Properties of transfer function.

$$\overline{A(\lambda)} = A(-\lambda), \text{ real-valued data}$$

$$A(\lambda+2\pi) = A(\lambda), \qquad \exp\{i2\pi\}=1$$

fundamental domain for λ: $[0,\pi]$

Vector case. **a** is s by r

$\mathbf{Y}(t) = \Sigma\, \mathbf{a}(u)\mathbf{X}(t-u)$ $\mathbf{A}(\lambda) = \Sigma\, \mathbf{a}(u) \exp\{-i\lambda u\}$

Continuous.

$Y(t) = \int a(u)X(t-u)du$ $A(\lambda) = \int a(u) \exp\{-i\lambda u\}\, du$

Spatial.

$Y(x,y) = \Sigma\, a(u,v)X(x-u,y-v)$ $A(\lambda,\mu)=\Sigma_{u,v}\, a(u,v) \exp\{-i(\lambda u+\mu v)\}$

Point process.

$Y(t) = \int a(u)dN(t-u) = \Sigma\, a(t-\tau_j)$ $A(\lambda) = \int a(u) \exp\{-i\lambda u\}du$

Algebra (of manipulating linear time invariant operators).

Linear combination $A[X] + B[X]$

$$A(\lambda) + B(\lambda) \equiv a(t) + b(t)$$

successive application $B[A[X]]$

$$B(\lambda)A(\lambda) \equiv b * a(t)$$

inverse $A^{-1}[X]$

$$B(\lambda) = A(\lambda)^{-1}$$

Impulse response.

Dirac delta $\delta(u)$, u in R

Kronecker delta δ_u = 1 if u=0, = 0 otherwise

$A[\delta](t) = a(t)$ *impulse response*

$a(u) = 0, t<0$ *realizable*

Examples.

Running mean of order 2M+1.

$Y(t) = \Sigma^{M}_{-M} X(t+u)/(2M+1)$

$$A(\lambda) = [\sin(2M+1)\lambda/2]/[(2M+1)\sin(\lambda/2)]$$

Difference

$Y(t) = X(t) - X(t-1)$

$A(\lambda) = 2i \sin(\lambda/2) \exp\{-i\lambda/2\}$

$|A(\lambda)|$

Lowpass filter cutoff Ω a smoother

Transfer function $A(\lambda) = 1 \quad |\lambda| \le \Omega$

$Y(t) = \Sigma\, a(u)\, X(t-u)$ t,u in Z

$A(\lambda) = \Sigma\, a(u) \exp\{-i \lambda u) \quad -\pi < \lambda \le \pi$

$a(u) = \int \exp\{iu\lambda\}A(\lambda)d\lambda / 2\pi$

$= \int_{|lamda|<Omega} \exp\{i u\lambda\}d\lambda/2\pi$

$= \Omega/\pi \quad u=0$

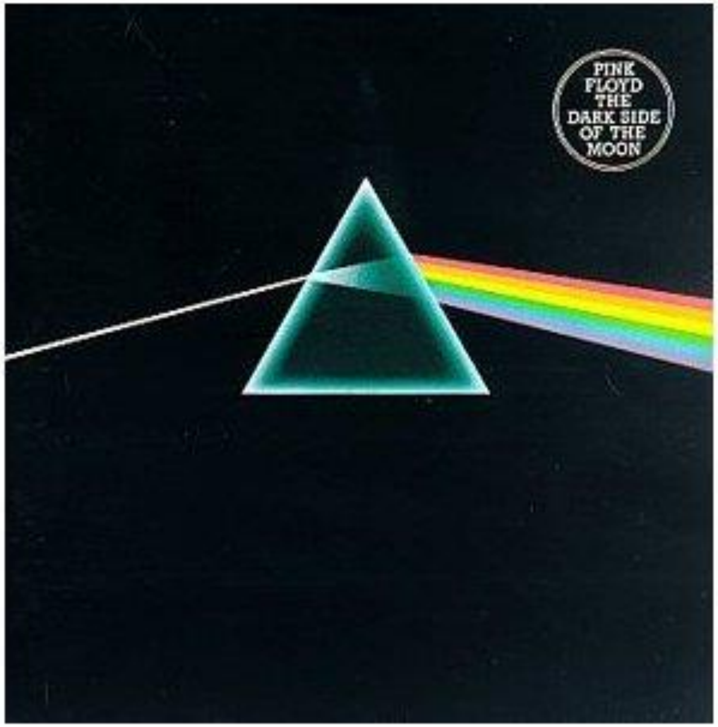
$= \sin \Omega u/\pi u \quad u \ne 0$

Bandpass filter central frequency λ_0 bandwidth 2δ

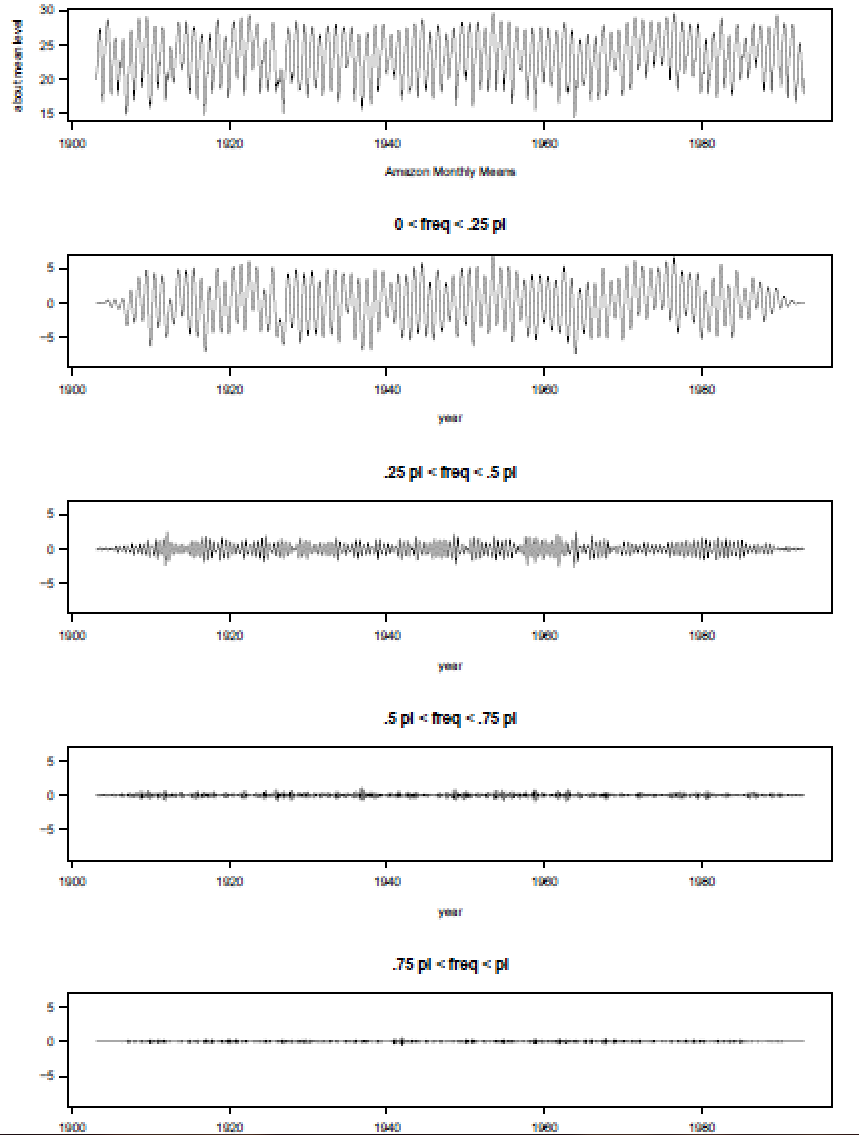
$A(\lambda)$ = 1 for λ in (,)
= 0 otherwise

Bank of bandpass filters

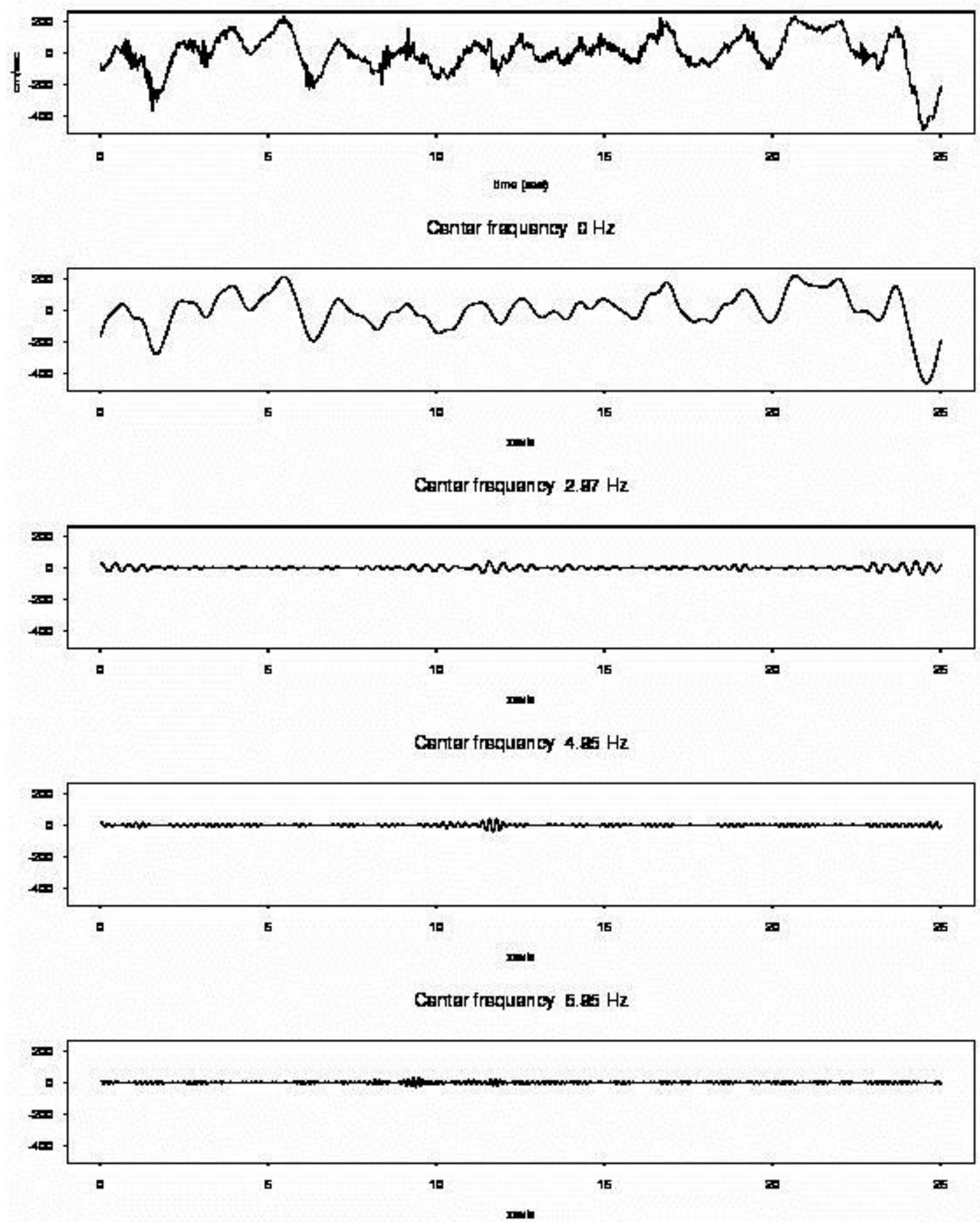
Decomposes a series, additively

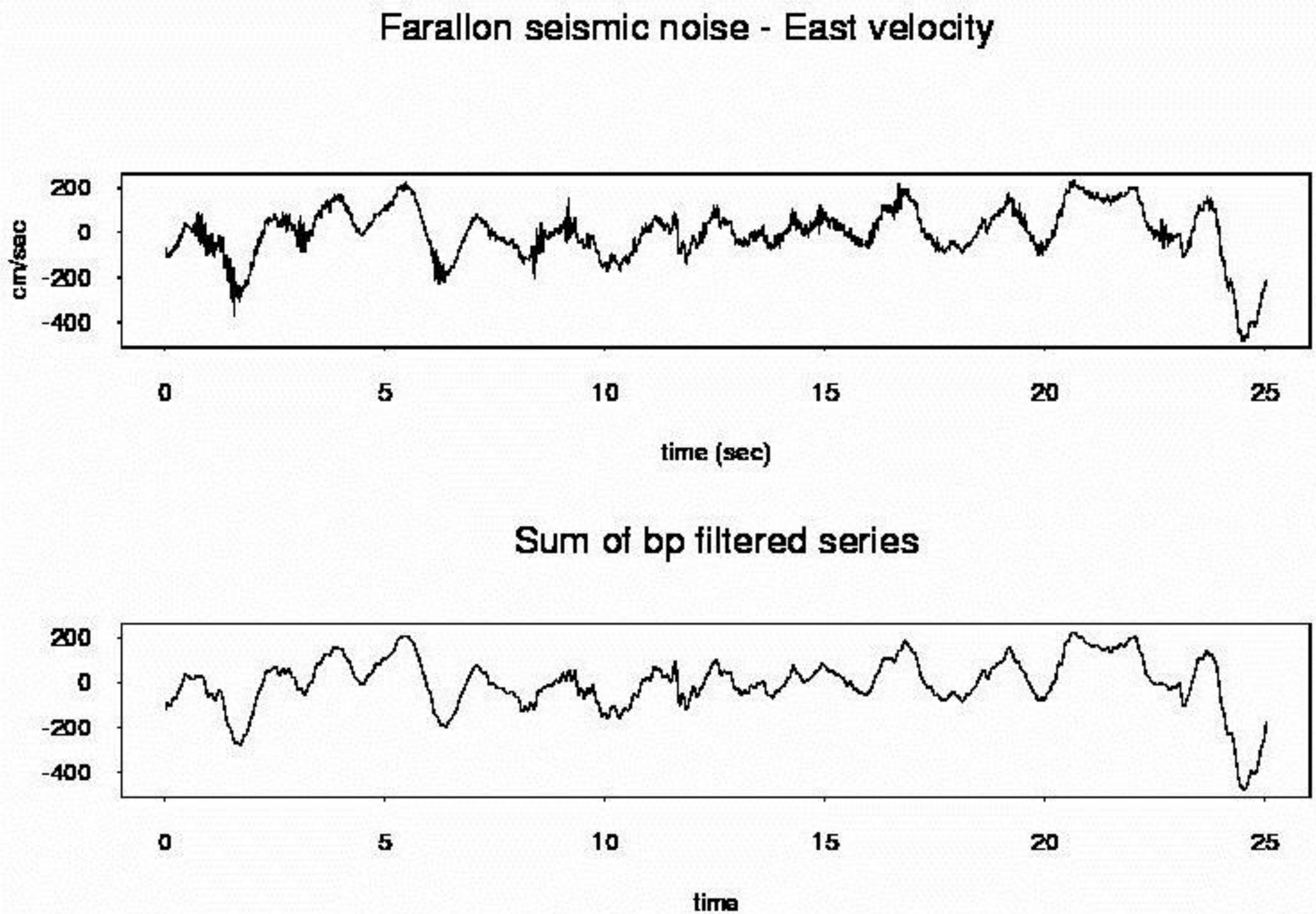

PINK
FLOYD
THE
DARK SIDE
OF THE
MOON

Amazon River at Manaus



Farallon seismic noise - East velocity




Farallon seismic noise - East velocity
200
0
-200
-400
cm/sec
0
5
10
15
20
25
time (sec)
Sum of bp filtered series
200
0
-200
-400
0
5
10
15
20
25
time

Nonlinear.

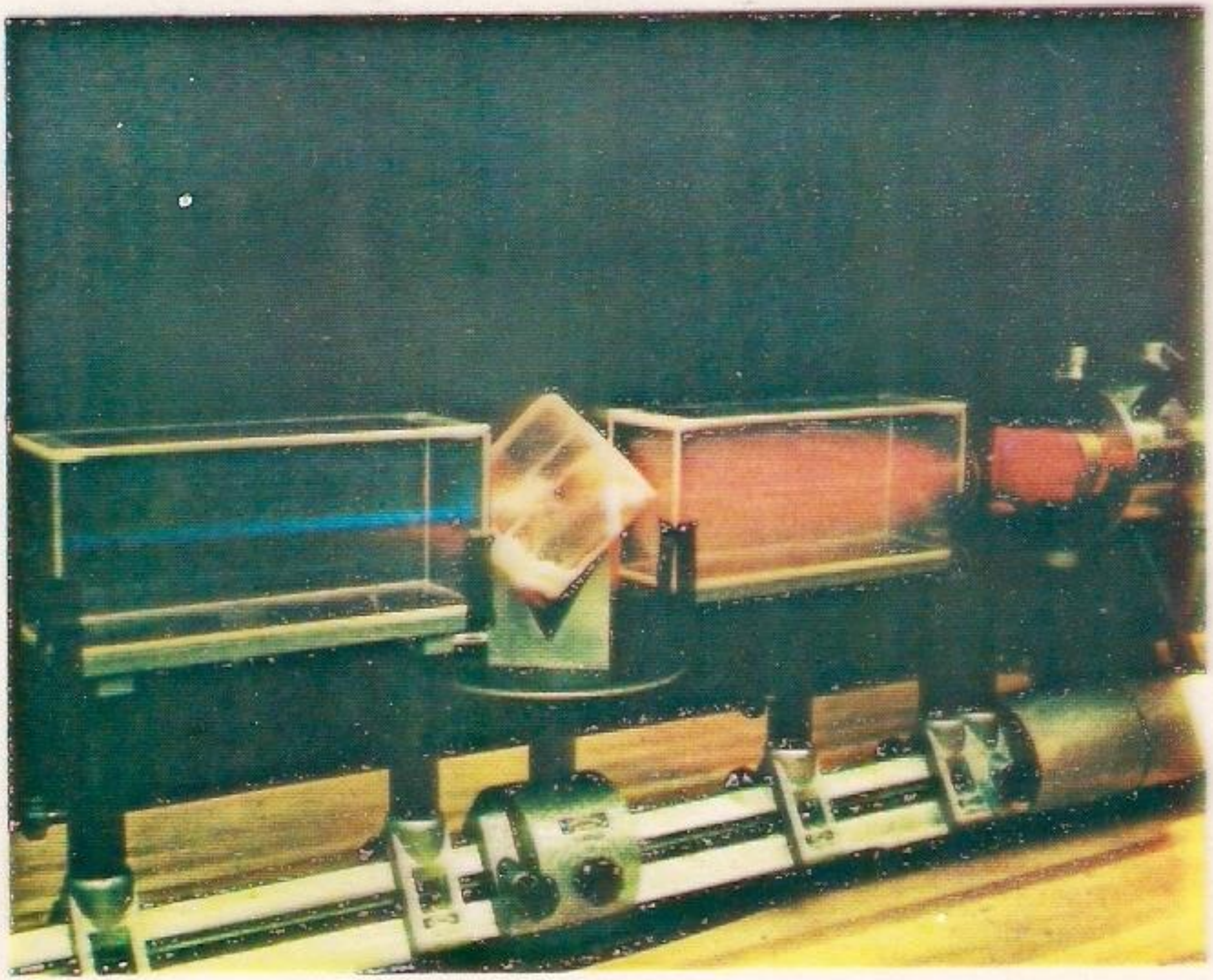
Quadratic instantaneous.

$$Y(t) = X(t)^2$$

$$X(t) = \cos \lambda t$$

$$Y(t) = (1 + \cos 2\lambda t)/2$$

Yariv "Quantum Electronics"



Laser beam enters a crystal of ammonium dihydrogen phosphate as red light and emerges as blue—the second harmonic. Courtesy of R. W. Terhune.

Quadratic, time lags (Volterra)

$$Y(t) = \Sigma\, a(u)X(t-u) + \Sigma_{u,v}\, b(u,v)\, X(t-u)\, X(t-v)$$

quadratic transfer function

$$B(\lambda,\mu) = \Sigma_{u,v}\, b(u,v)\, \exp\{-i(\lambda u+\mu v)\}$$

Transforms.

Fourier.

$d^T(\lambda) = \Sigma_t\ Y(t) \exp\{-i\lambda t\}$ λ real

$= \int Y(t) \exp\{-i\lambda t\}dt$

Laplace.

$L^T(p) = \Sigma_{t\geq 0}\ Y(t) \exp\{-pt\}$ p complex

$= \int Y(t) \exp\{-pt\}dt$

z.

$Z^T(z) = \Sigma_{t\geq 0}\ Y(t)z^t$ z complex

Hilbert.

$Y(t) = \Sigma_u\ a(u)\ X(t-u)$

$a(u) = 2/\pi u$, u odd = 0, u even

$\int_{u \neq t} X(u)/(t-u)du$

$\cos \lambda t \Longrightarrow \sin \lambda t$

Radon. Y(x,y)

$R(\alpha,\beta) = \int Y(x,\alpha+\beta x)dx$

Short-time/running Fourier.

$Y(t) = \int w(t-u)\ \exp\{-i\lambda u)X(u)du$

Gabor: $w(u) = \exp\{-ru^{2}\}$

Wavelet.

$$W(u,a) = \int \overline{w}((t-u)/a)Y(t)dt / \sqrt{|a|}$$

eg. $w(t) = w_0(t) \exp\{-i\lambda t\}$

Walsh.

$$\int \psi_\lambda (t) Y(t) dt$$

$\psi_n(t)$: Walsh function $\quad \psi_s(t) = \psi_{[s]}(t)\psi_{[t]}(s)$

Chirplet.

$$C(\lambda,\mu) = \int Y(t)\exp\{-i(\lambda+\mu t)t\}dt$$

Use of $A(\lambda,\mu)$.

Suppose $X(x,y) \approx \Sigma_{j,k}\ \alpha_{jk} \exp\{i(\lambda_j x + \mu_k y)\}$

$Y(x,y) = A[X](x,y)$

$\approx \Sigma_{j,k}\ A(\lambda_j,\mu_k)\ \alpha_{jk} \exp\{i(\lambda_j x + \mu_k y)\}$

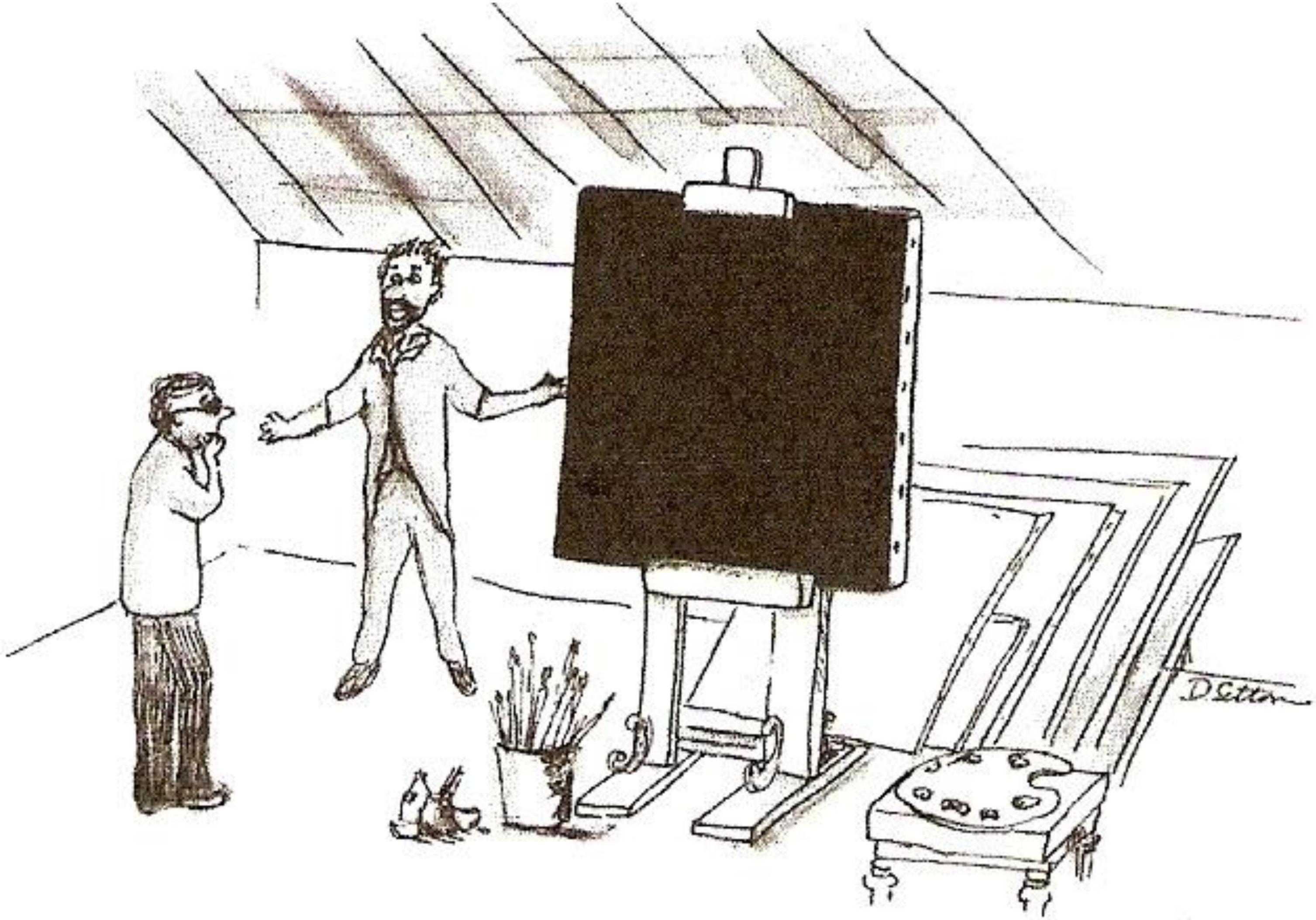
e.g. If $A(\lambda,\mu) = 1$, $\quad |\lambda \pm \lambda_0|, |\mu \pm \mu_0| \leq \Delta$

$= 0$ $\quad$ otherwise

Y(x,y) contains only these terms

Repeated xeroxing

Chapters 2,3 in D. R. Brillinger "Time Series: Data Analysis and Theory". SIAM paperback



"I call it 'Astronomy Beyond the Visible-light Spectrum.'"