State space model

$$\underset{\sim}{Y}_t = \underset{\sim}{G}_t \underset{\sim}{X}_t + \underset{\sim}{W}_t$$ observation eqⁿ measurement

$$\underset{\sim}{X}_{t+1} = \underset{\sim}{F}_t \underset{\sim}{X}_t + \underset{\sim}{V}_t \quad (*)$$ state eqⁿ transition

$\{\underset{\sim}{W}_t\}, \{\underset{\sim}{V}_t\} \sim WN$, mean $\underset{\sim}{0}$; $\underset{\sim}{R}_t$, $\underset{\sim}{Q}_t$

$$E\, \underset{\sim}{W}_t \underset{\sim}{V}_s' = \underset{\sim}{0}$$

$\underset{\sim}{X}_t$ all pertinent past

Control term: RHS $(*)$ $+ \underset{\sim}{H}_t \underset{\sim}{u}_t$

Initial conditions

Data: $\underset{\sim}{Y}_t, \underset{\sim}{u}_t \quad t = 1, \ldots, n$

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Uses

Prediction

$$\hat{\underset{\sim}{X}}_t = P_{t-1}(\underset{\sim}{X}_t)$$
$$= P(\underset{\sim}{X}_t \mid \underset{\sim}{Y}_0, \ldots, \underset{\sim}{Y}_{t-1})$$

$$\hat{Y}_t = \underset{\sim}{G}_t \hat{\underset{\sim}{X}}_t$$

Filtering

$$P_t(\underset{\sim}{X}_t) = P(\underset{\sim}{X}_t \mid \underset{\sim}{Y}_0, \ldots, \underset{\sim}{Y}_{t-1})$$

Smoothing

$$P_m(\underset{\sim}{X}_t) = P(\underset{\sim}{X}_t \mid \underset{\sim}{Y}_0, \ldots, \underset{\sim}{Y}_m)$$

$$m > t$$

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Recursive, constant dimension computations

Missing values

Likelihood analysis

Replaces Box-Jenkins
Aggregation (annual, monthly data)
Universal approach
Irregularly spaced
Measurement error
Structural breaks
Outliers

...

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Setting up ARMA(p, q)

$$\varphi(B) Y_t = \theta(B) Z_t$$

$$r = \max(p, q+1)$$

$$\theta_0 = 1, \quad \theta_j = 0 \quad j > q$$

$$\varphi_j = 0 \quad j > p$$

$$\varphi(B) U_t = Z_t$$

$$Y_t = \theta(B) \varphi(B)^{-1} Z_t = \theta(B) U_t$$

(assump)

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Set $W_t = 0$

$$\underset{\sim}{X}_t = \begin{bmatrix} U_{t-r+1} \\ U_{t-r+2} \\ \vdots \\ U_t \end{bmatrix}$$

$$\underset{\sim}{G}_t = [\,\theta_{r-1} \quad \theta_{r-2} \quad \cdots \quad \theta_0\,] = \underset{\sim}{G}$$

$$\underset{\sim}{V}_t = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} Z_{t+1}$$

$$\underset{\sim}{F}_t = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & & 1 \\ \varphi_r & \varphi_{r-1} & \varphi_{r-2} & \cdots & \varphi_1 \end{bmatrix} = \underset{\sim}{F}$$

Another form

$$Y_t = [1 \;\; \underset{\sim}{0}'_{r-1}] \underset{\sim}{X}_t$$

$$\underset{\sim}{X}_{t+1} = \underset{\sim}{F} \underset{\sim}{X}_t + \underset{\sim}{\Psi} Z_t$$

$$\underset{\sim}{F} = \left[\begin{array}{c|c} -\varphi_1 & \\ \vdots & \underset{\sim}{I}_{r-1} \\ -\varphi_{r-1} & \\ \hline -\varphi_r & \underset{\sim}{0}'_{r-1} \end{array}\right] \qquad \underset{\sim}{\Psi} = \begin{bmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{r-1} \end{bmatrix}$$

$$Y_t = X_{1t}$$

$$X_{r,t} = -\varphi_r Y_{t-1} + \theta_{r-1} Z_t$$

$$X_{r-1,t} = -\varphi_{r-1} Y_{t-1} + X_{r,t-1} + \theta_{r-2} Z_t$$

$$\vdots$$

$$X_{1,t} = -\varphi_1 Y_{t-1} - \varphi_2 Y_{t-2} - \cdots + Z_t + \theta_1 Z_{t-1} + \cdots$$

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Random walk + noise

$Y_t = M_t + W_t$ $\{W_t\}: WN(0,\sigma_w^2)$

$M_{t+1} = M_t + V_t$ $\{V_t\}: WN(0,\sigma_v^2)$

$X_t = M_t$

$D_t = \nabla Y_t = \nabla M_t + \nabla W_t$

$= V_{t-1} + W_t - W_{t-1}$

D_t has $ED_t = 0$

$\text{var} D_t = \sigma_v^2 + 2\sigma_w^2$

$\text{cov}\{D_t, D_{t-1}\} = -\sigma_w^2$

$\text{cov}\{D_t, D_{t-h}\} = 0$ $h > 1$

i.e. is MA(1)

So Y_t is ARIMA(0,1,1)

t Mar 03

Random walk + noise.

$$Y_t = X_t + W_t \qquad \{W_t\} \sim WN(0, \sigma_w^2)$$

$$X_{t+1} = X_t + V_t \qquad \{V_t\} \sim WN(0, \sigma_v^2)$$

$$\hat{Y}_{t+1} = P_t Y_{t+1} = \hat{X}_t + \frac{\Theta_t}{\Delta_t}(Y_t - \hat{Y}_t)$$

$$= (1 - a_t)\hat{Y}_t + a_t Y_t$$

$$a_t = \frac{\Theta_t}{\Delta_t}$$

Random walk + drift.

$$Y_t = M_t + W_t$$

$$M_t = M_{t-1} + B_{t-1} + V_{t-1}$$

$$\underset{\sim}{X}_t = \begin{bmatrix} M_t \\ B_t \end{bmatrix}$$

$$Y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \underset{\sim}{X}_t + W_t$$

$$\underset{\sim}{X}_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underset{\sim}{X}_t + \underset{\sim}{V}_t$$

Eg.

$$B_t = B$$

$$\underset{\sim}{X}_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underset{\sim}{X}_t + \begin{bmatrix} V_t \\ 0 \end{bmatrix}$$

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Innovations. $\underset{\sim}{I}_0 = \underset{\sim}{Y}_0$

$$\underset{\sim}{I}_t = \underset{\sim}{Y}_t - P_{t-1} \underset{\sim}{Y}_t$$

The new information

$$E\{\underset{\sim}{I}_t \mid \ldots, \underset{\sim}{Y}_{t-1}\} = 0 = E\{\underset{\sim}{I}_t\}$$

$$\mathrm{Var}\{\underset{\sim}{I}_t \mid \ldots, Y_{t-1}\} = \underset{\sim}{\Delta}_t$$

$$= \underset{\sim}{G}_t \underset{\sim}{\Omega}_t \underset{\sim}{G}_t' + \underset{\sim}{R}_t$$

$$\mathrm{Cov}\{\underset{\sim}{X}_{t+1}, \underset{\sim}{I}_t\} = \underset{\sim}{\Theta}_t$$

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Likelihood.