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General state space modelling

parameter-driven

Suppose

$$\underset{\sim}{X}_{t+1} \mid \underset{\sim}{X}_t, \underset{\sim}{X}^{(t-1)}, \underset{\sim}{Y}^{(t-1)} \quad \text{has density}$$

$$q(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{x}_t)$$

<u>state</u>

Suppose

$$\underset{\sim}{Y}_t \mid \underset{\sim}{X}^{(t)}, \underset{\sim}{Y}^{(t-1)} \quad \text{has density}$$

$$r(\underset{\sim}{y}_t \mid \underset{\sim}{x}_t)$$

<u>observation</u>

cp.

$$\underset{\sim}{X}_{t+1} = \underset{\sim}{F}\underset{\sim}{X}_t + \underset{\sim}{V}_t$$

$$\underset{\sim}{Y}_t = \underset{\sim}{G}\underset{\sim}{X}_t + \underset{\sim}{W}_t$$

$$q \sim N(\underset{\sim}{F}\underset{\sim}{X}_t, \underset{\sim}{Q})$$

$$r \sim N(\underset{\sim}{G}\underset{\sim}{X}_t, \underset{\sim}{R})$$

Other cases

Nonlinear

$$\underset{\sim}{X}_{t+1} = \underset{\sim}{f}(\underset{\sim}{X}_t) + \underset{\sim}{V}_t = \underset{\sim}{f}(\underset{\sim}{X}_t, \underset{\sim}{V}_t)$$

$$\underset{\sim}{Y}_t = \underset{\sim}{h}(\underset{\sim}{X}_t) + \underset{\sim}{W}_t = \underset{\sim}{h}(\underset{\sim}{X}_t, \underset{\sim}{W}_t)$$

Discrete

$$\underset{\sim}{X}_{t+1} = \underset{\sim}{F}\,\underset{\sim}{X}_t + \underset{\sim}{V}_t$$

$$\underset{\sim}{Y}_t \sim \mathrm{Dist}(\underset{\sim}{X}_t)$$

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The recursions

Prediction (one-step)

$p(\mathbf{x}_{t+1} \mid \mathbf{y}^{(t)})$

$$= \int p(\mathbf{x}_{t+1}, \mathbf{x}_t \mid \mathbf{y}^{(t)})\, d\mathbf{x}_t$$

$$= \int p(\mathbf{x}_{t+1} \mid \mathbf{x}_t, \mathbf{y}^{(t)})\, p(\mathbf{x}_t \mid \mathbf{y}^{(t)})\, d\mathbf{x}_t$$

$$= \int q(\mathbf{x}_{t+1} \mid \mathbf{x}_t)\, p(\mathbf{x}_t \mid \mathbf{y}^{(t)})\, d\mathbf{x}_t \qquad (**)$$

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Filtering.

$$p(\underset{\sim}{x}_t \mid \underset{\sim}{y}^{(t)})$$

$$= p(\underset{\sim}{x}_t \mid \underset{\sim}{y}_t, \underset{\sim}{y}^{(t-1)})$$

$$= \frac{p(\underset{\sim}{y}_t \mid \underset{\sim}{x}_t, \underset{\sim}{y}^{(t-1)})\, p(\underset{\sim}{x}_t \mid \underset{\sim}{y}^{(t-1)})}{p(\underset{\sim}{y}_t \mid \underset{\sim}{y}^{(t-1)})}$$

$$= \frac{r(\underset{\sim}{y}_t \mid \underset{\sim}{x}_t)\, p(\underset{\sim}{x}_t \mid \underset{\sim}{y}^{(t-1)})}{p(\underset{\sim}{y}_t \mid \underset{\sim}{y}^{(t-1)})}$$

$$p(\underset{\sim}{y}_t \mid \underset{\sim}{y}^{(t-1)}) = \int r(\underset{\sim}{y}_t \mid \underset{\sim}{x}_t)\, p(\underset{\sim}{x}_t \mid \underset{\sim}{y}^{(t-1)})\, d\underset{\sim}{x}_t \quad (*)$$

Smoothing.

$$p(\underset{\sim}{x}_t \mid \underset{\sim}{y}^{(n)})$$

$$= \int p(\underset{\sim}{x}_t, \underset{\sim}{x}_{t+1} \mid \underset{\sim}{y}^{(n)})\, d\underset{\sim}{x}_{t+1}$$

$$= \int p(\underset{\sim}{x}_{t+1}, \underset{\sim}{y}^{(n)})\, p(\underset{\sim}{x}_t \mid \underset{\sim}{x}_{t+1}, \underset{\sim}{y}^{(t)})\, d\underset{\sim}{x}_{t+1}$$

$$= p(\underset{\sim}{x}_t \mid \underset{\sim}{y}^{(t)}) \int \frac{p(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{y}^{(n)})\, p(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{x}_t, \underset{\sim}{y}^{(t)})}{p(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{y}^{(t)})}\, d\underset{\sim}{x}_{t+1}$$

$$= p(\underset{\sim}{x}_t \mid \underset{\sim}{y}^{(t)}) \int \frac{p(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{y}^{(n)})\, g(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{x}_t)}{p(\underset{\sim}{x}_{t+1} \mid \underset{\sim}{y}^{(t)})}\, d\underset{\sim}{x}_{t+1}$$

Likelihood $\displaystyle\prod_{t=1}^{n} p(\underset{\sim}{y}_t \mid \underset{\sim}{y}^{(t-1)})$

see (*)

Numerical synthesis.

Need to evaluate the integrals & likelihood

- numerical integration
- Monte Carlo

See G. Kitagawa & W. Gersch (1996) Smoothness Priors Analysis of Time Series. Lecture Notes in Statistics 116. Springer.