(i) 13 March 03

Maximum Likelihood

$\underset{\sim}{X} \in R^n$, $\underset{\sim}{\theta} \in \Theta \subseteq R^m$

p.d.f. $p(\underset{\sim}{x}; \underset{\sim}{\theta})$

$L(\underset{\sim}{\theta}) = p(\underset{\sim}{x}; \underset{\sim}{\theta})$ $\underset{\sim}{x}$ data

$$\hat{\underset{\sim}{\theta}} = \arg \max_{\underset{\sim}{\theta} \in \Theta} p(\underset{\sim}{x}; \underset{\sim}{\theta})$$

$$l(\underset{\sim}{\theta}) = \log p(\underset{\sim}{x}; \underset{\sim}{\theta})$$

I.i.d. case $X \in R$

$$\sqrt{n}\,(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta}_0) \xrightarrow{d} N_m(\underset{\sim}{0}, \underset{\sim}{I}(\underset{\sim}{\theta}_0)^{-1})$$

$$\underset{\sim}{I}(\underset{\sim}{\theta}) = E_\theta \left\{ \frac{\partial \log p(X; \underset{\sim}{\theta})}{\partial \underset{\sim}{\theta}} \frac{\partial \log p(X; \underset{\sim}{\theta})'}{\partial \underset{\sim}{\theta}} \right\}$$

$$\underset{\sim}{I}(\hat{\underset{\sim}{\theta}})$$

CIs , H_0

Dependent case.

Take $n=1$

$$p(x_1,\dots,x_n;\underset{\sim}{\theta}) = p(\underset{\sim}{x};\underset{\sim}{\theta})$$

$$\hat{\underset{\sim}{\theta}} \sim N_m\left(\underset{\sim}{\theta}_0, \left[E_{\theta_0} \frac{\partial \log p(\underset{\sim}{X};\theta)}{\partial \underset{\sim}{\theta}} \frac{\partial \log p(\underset{\sim}{X};\theta)'}{\partial \underset{\sim}{\theta}}\right]^{-1}\right)$$

$$p(x_1,\dots,x_n;\underset{\sim}{\theta}) = p(x_1;\underset{\sim}{\theta})\, p(x_2|x_1;\underset{\sim}{\theta}) \dots$$
$$p(x_n|x_{n-1},x_{n-2},\dots,x_1;\underset{\sim}{\theta})$$

$$\log p(\underset{\sim}{X};\underset{\sim}{\theta})$$
$$= \log p(X_1;\underset{\sim}{\theta}) + \log p(X_2|X_1;\underset{\sim}{\theta})$$
$$+ \dots + \log p(X_n|X_{n-1},\dots,X_1;\underset{\sim}{\theta})$$

$\exists$ LLNs, CLTs

Gaussian likelihood <u>TSTM</u> §10.8

Simulation, initial values

13 March 03

Random walk plus noise.

$$X_{t+1} = X_t + V_t \qquad WN(0, \sigma_v^2)$$

$$\perp$$

$$Y_t = X_t + W_t \qquad WN(0, \sigma_w^2)$$

$$Y_0 = 1, \quad R = \sigma_w^2, \quad Q = \sigma_v^2$$

$$F_t = 1, \quad G_t = 1$$

$$\hat{X}_{t+1} = \hat{X}_t + \Theta_t \Delta_t^{-1}(Y_t - \hat{X}_t)$$

$$\Theta_t = \Omega_t$$

$$\Delta_t = \Omega_t + \sigma_w^2$$

$$\Omega_{t+1} = \Omega_t + \sigma_v^2 - \Theta_t^2 \Delta_t^{-1}$$

$$= \Omega_t + \sigma_v^2 - \Omega_t^2 / (\Omega_t + \sigma_w^2)$$

$$= \sigma_v^2 + \Omega_t \sigma_w^2 / (\Omega_t + \sigma_w^2)$$

$$\hat{Y}_{t+1} = \hat{X}_{t+1}$$

$$a_t = \Theta_t \Delta_t^{-1} = \Omega_t / (\Omega_t + \sigma_w^2)$$

<u>Kalman gain</u>

13 March 03

$$\hat{X}_{t+1} = (1-a_t)\hat{X}_t + a_t Y_t$$

$$\hat{Y}_{t+1} = (1-a_t)\hat{Y}_t + a_t Y_t$$

Example:

If $\sigma_v^2 = 0$, $X_t = \mu$

If take $\hat{X}_1 = Y_1$, $\Omega_1 = \sigma_w^2$

$$\Omega_{t+1} = \Omega_t \sigma_w^2 / (\Omega_t + \sigma_w^2)$$

$$\Gamma_{t+1} = \Omega_{t+1}/\sigma_w^2 = \Gamma_t / (\Gamma_t + 1)$$

$$\Gamma_t = 1/t$$

$$a_t = (t-1)/t$$

$$\hat{X}_{t+1} = \frac{1}{t}\hat{X}_t + \frac{t-1}{t} Y_t$$

$$\bar{Y}_{t+1} = \frac{1}{t+1}\bar{Y}_t + \frac{t}{t+1} Y_{t+1}$$

13 March 03

h-step prediction.

$h > 0$

$$P_t Y_{t+h} = P_t \left(G_{t+h} X_{t+h} + W_{t+h} \right)$$

$$= G_{t+h} P_t X_{t+h}$$

$$P_t X_{t+h} = P_t \left(F_{t+h-1} X_{t+h-1} + V_{t+h-1} \right)$$

$$= F_{t+h-1} P_t X_{t+h-1}$$

$$\vdots$$

$$= F_{t+h-1} F_{t+h-2} \cdots F_{t+1} P_t X_{t+1}$$