Statistics 248 - 28 Jan. 04 - D. R. Brillinger

Some Chapter 1 and 2 Concepts and Notation

System - object in which variables of different kinds interact and produce observable signals

Signal

Output - observable signal of interest

Input - external signal that can be manipulated

Dynamic system - current output depends on present and past external stimuli

Time series - "output of dynamic system whose external stimuli are not observed"

System
  Input, $u(t)$ Output, $y(t)$

Linear

Causal

Linear time-invariant **causal** *system*

$$y(t) = \sum_{k=1}^{\infty} g(k)\ u(t-k)$$

$$y(t) = \int_0^{\infty} g(\tau)\ u(t-\tau)\ d\tau$$

Impulse response, $g(\tau)$

Sampling interval, T $t_k = kT$

Disturbance
  $v(t)$ unmeasured, $w(t)$ measured

E x, Var x

White noise, e(t)

$E\ e(t) = 0,\ Var\ e(t) = \lambda$

Stochastic process

Realization

$\{e(t)\},\ \{v(t)\},\ v(t) = \sum_{k=0}^{\infty} h(k)\ e(t-k)$

$E\ v(t) = 0$

Second-order properties

Covariance function, $E\ v(t) = 0$

$R_v(\tau) = E\ v(t)\ v(t-\tau) = cov\{v(t),\ v(t-\tau)\}$

Stationary

$E\ v(t) = 0$

$R_v(\tau) = E\ v(t)\ v(t-\tau) = \lambda \sum_{k=0}^{\infty} h(k)\ h(k-\tau)$

Forward & backward shift operators, q & q^{-1}

$q\ u(t) = u(t+1)$ & $q^{-1}\ u(t) = u(t-1)$

Transfer function

$y(t) = \sum_{k=1}^{\infty} g(k)\ u(t-k)$

$= G(q)u(t)$

$$G(q) = \sum_{k=1}^{\infty} g(k)\, q^{-k}$$

Linear system with additive disturbance

$$y(t) = \sum_{k=1}^{\infty} g(k)u(t-k) + v(t)$$

$$= G(q)u(t) + H(q)e(t)$$

System G ≡ Filter G

Stable

$$\sum |g(k)| < \infty$$

Cosinusoid input

$$u(t) = \cos \omega t = \mathrm{Re}\, e^{i\omega t}$$

Frequency response

$$y(t) = |G(e^{i\omega t})| \cos(\omega t + \varphi),\ \varphi = \arg G(e^{i\omega t})$$

DFT

$$U_N(\omega) = \sum_{k=1}^{N} u(t)e^{-i\omega t}/\sqrt{N}$$

$$u(t) = \sum_{k=1}^{N} U_N(2\pi k/N)e^{i2\pi t/N}/\sqrt{N}$$

$$U_N(\omega+2\pi) = U_N(\omega)$$

$$\mathrm{conj}\ U_N(\omega) = U_N(-\omega)$$

Transformation of FT

$$s(t) = \sum_{k=1}^{\infty} g(k)\,w(t-k)$$

$$= G(q)\,w(t)$$

$$S_n(\omega) \approx G(e^{i\omega})\,W_n(\omega)$$

Periodogram, $|U_n(\omega)|^2$

Transformation of periodogram

$$|S_n(\omega)|^2 \approx |G(e^{i\omega})|^2\,|W_n(\omega)|^2$$

Signal spectrum

$\lim_{n\to\infty} |S_n(\omega)|^2$ if it exists (GHA)

(Power) spectrum

$$\Phi_s(\omega) = \sum_{\tau=-\infty}^{\infty} R_s(\tau)\,e^{-i\omega\tau}$$

Important properties

$$R_s(\tau) = \int_{-\pi}^{\pi} e^{i\omega\tau}\,\Phi_s(\omega)\,d\omega\ /2\pi$$

Examples - white noise, cosinusoid, ...

Cross spectrum

$$\Phi_{sw}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{sw}(\tau)\,e^{-i\omega\tau}$$

cospectrum, quadrature spectrum, phase spectrum, amplitude spectrum

Periodogram is asymptotically unbiased

$E\ |S_n(\omega)|^2 \rightarrow \Phi_s(\omega)$

Algebra of filters & spectra

$s(t) = G(q)w(t)$

$\Phi_s(\omega) = |G(e^{i\omega})|^2\ \Phi_w(\omega)$

Ordinary statistics form

Examples - lowpass, bandpass

$\Phi_{sw}(\omega) = G(e^{i\omega})\ \Phi_w(\omega)$

Spectral factorization, $v(t) = R(q)e(t)$

$\Phi_v(\omega) = \lambda\ |R(e^{i\omega})|^2$

ARMA process

R(q) is rational function

ARMA (m, n)

$v(t) + a_1 v(t-1) + \ldots + a_m v(t-m)$

$= e(t) + c_1 e(t-1) + \ldots + c_n e(t-n)$

AR (m)

$$v(t) + a_1 v(t-1) + \ldots + a_m v(t-m) = e(t)$$

MA (n)

$$v(t) = e(t) + c_1 e(t-1) + \ldots + c_n(t-n)$$

$$v(t) = R(q)e(t) \text{ with}$$

$$R(q) = C(q)/A(q)$$

Stationary?

Second-order statistics

$$\Phi_v(\omega) = \lambda \; |R(e^{i\omega})|^2 = \lambda \; |A(e^{i\omega})|^2/|C(e^{i\omega})|^2$$

$$R_y(\tau) = \int_{-\pi}^{\pi} e^{i\omega\tau} \Phi_y(\omega) d\omega \; /2\pi$$

Ergodicity

$$\lim_{n\to\infty} \sum_1^n s(t)/N = Es(t) \quad \text{a.s.}$$

$$\lim_{n\to\infty} \sum_{t=1}^n s(t)s(t-\tau)/N = Es(t)s(t-\tau) \quad \text{a.s.}$$

Quasi-stationary signal

$$Es(t) = m_s(t)$$

$$Es(t)s(r) = R_s(t,r)$$

$$\lim_{n\to\infty} \sum_{t=1}^n R_s(t,t-\tau)/N = R_s(\tau) = \bar{E}\, s(t)s(t-\tau)$$

Multivariable system

$$y(t) = G(q)u(t) + H(q)w(t)$$

$$\Phi_y(\omega) = G(e^{i\omega})\Phi_y(\omega)G^T(e^{-i\omega}) + H(e^{i\omega})\Lambda H^T(e^{-i\omega})$$

Transfer-function model

$$y(t) = G(q)u(t) + v(t)$$

$$= \sum_k g(k)u(t-k) + v(t)$$

u and v independent, v unmeasured

$$\Phi_y(\omega) = |G(e^{i\omega})|^2\Phi_u(\omega) + \Phi_v(\omega)$$

$$\Phi_{yu}(\omega) = G(e^{i\omega})\Phi_u(\omega)$$

Summary

$$y(t) = G(q)u(t) + H(q)e(t)$$

$$G(q) = \sum_{k=1}^{\infty} g(k)\ q^{-k}$$

$$H(q) = 1 + \sum_{k=1}^{\infty} h(k)\ q^{-k}$$

Filters may be two-sided as well

Problems

State space description

$$x(t+1) = F(t)x(t) + w(t)$$

$$y(t) = H(t)x(t) + v(t)$$

$x(t)$: state vector

Can put ARMA in this form

Appendix

Proofs