Statistics 248 - Homework 7x (D. R. Brillinger)

In class we gave the relation

$$cum\{Y_1, \cdots, Y_K\} = \sum_{\nu} (-1)^{p-1}(p-1)!E\{\prod_{k\varepsilon\nu_1} Y_k\} \cdots E\{\prod_{k\varepsilon\nu_p} Y_k\}$$

where $\sum$ is over all partitions $\nu = (\nu_1, \ldots, \nu_p)$, $p = 1,...,K$ of $(1,...,K)$.

The inverse of this is

$$E\{Y_1 \cdots Y_K\} = \sum_{\nu} cum\{Y_{\nu_1}\} \cdots cum\{Y_{\nu_p}\}$$

which may be useful to you in solving the problem below.

Problem.

For some integer K, let $(Y_1^{(j)}, \cdots, Y_K^{(j)})$, $j = 1,...,K$ be i.i.d. . Set

$$Z_l = \sum_{j=1}^{K} Y_l^{(j)} \exp\{i\,2\pi j/K\}, \quad l = 1,...,K$$

a) Prove that for $(l_1, \ldots, l_k)$ a subset of $(1,...,K)$

$$cum\{Z_{l_1}, \cdots, Z_{l_k}\} = \sum_{j=1}^{K} \exp\{i\,2\pi jk/K\} cum\{Y_{l_1}, \cdots, Y_{l_k}\}$$

$$= 0 \qquad k < K$$
$$= K cum\{Y_{l_1}, \cdots, Y_{l_k}\} \qquad k = K$$

and then b) prove

$$cum\{Y_1, \cdots, Y_K\} = \frac{1}{K} E\{Z_1 \cdots Z_K\}$$

This last result can be useful when one wishes to represent a cumulant as a moment.

[Hint: you might first consider the case $K = 2$.]