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Spectral representation

$\{X_t\}$ covariance stationary

$\exists \; Z(\lambda) \ni$

$$X_t = \int_{-\pi}^{\pi} e^{it\lambda} \, dZ(\lambda), \quad t = 0, \pm 1, \ldots$$

with $\overline{Z(\lambda)} = Z(-\lambda)$,

$$\operatorname{cov}\{dZ(\lambda), dZ(\mu)\} = \eta(\lambda) f(\lambda) \, d\lambda \, d\mu$$

$$[\eta(\lambda) \, dF(\lambda) \, d\mu]$$

Cramér

$$\eta(\lambda) = \sum_j \delta(\lambda + 2\pi j)$$

Filtering

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k X_{t-k}$$

$$\Psi(e^{-i\lambda}) = \sum_{j=-\infty}^{\infty} \psi_j e^{-ij\lambda}$$

$$dZ_Y(\lambda) = \Psi(e^{-i\lambda})\, dZ_X(\lambda)$$

$$f_{YY}(\lambda) = |\Psi(e^{-i\lambda})|^2 f_{XX}(\lambda)$$

e.g. b. p. filter

$$Y(t) = \left(\int_{-\lambda_0-\Delta}^{-\lambda_0+\Delta} + \int_{\lambda_0-\Delta}^{\lambda_0+\Delta} \right) e^{it\lambda}\, dZ_X(\lambda)$$

$\exists$ algebra

③

Problem 3.5. $\{W_t\} \sim WN(0, \sigma_w^2)$

$$\perp$$

$\{X_t\} \sim ARMA(p, q)$

$r = \max(p, q)$

Consider $Y_t = X_t + W_t$

$$f_{YY}(\lambda) = f_{XX}(\lambda) + f_{WW}(\lambda)$$

$$= \frac{\sigma^2}{2\pi} \frac{|\theta_q(e^{-i\lambda})|^2}{|\varphi_p(e^{-i\lambda})|^2} + \frac{\sigma_w^2}{2\pi}$$

$$= \frac{1}{2\pi} \frac{\sigma^2 |\theta_q(e^{-i\lambda})|^2 + \sigma_w^2 |\varphi_p(e^{-i\lambda})|^2}{|\varphi_p(e^{-i\lambda})|^2}$$

$\sim ARMA(p, r)$

Aliasing.

Continuous time

$$X_t = \int_{-\infty}^{\infty} e^{it\lambda} \, d\tilde{Z}(\lambda), \quad -\infty < t < \infty$$

Discrete observations

$$X_t = \int_{-\pi}^{\pi} e^{it\lambda} \, dZ(\lambda), \quad \lambda = 0, \pm 1, \pm 2, \ldots$$

$$X_t = \sum_j \int_{2\pi j}^{2\pi(j+1)} e^{it\lambda} \, d\tilde{Z}(\lambda)$$

$$= \int_0^{2\pi} e^{it\lambda} \sum_j d\tilde{Z}(\lambda - 2\pi j)$$

$$dZ(\lambda) = \sum_j d\tilde{Z}(\lambda - 2\pi j)$$

$$f(\lambda) = \sum_j g(\lambda - 2\pi j)$$

$g(.)$ cts spectrum

Aliases $\left\{ \begin{matrix} \lambda + 2\pi j \\ -\lambda + 2\pi j \end{matrix}, \ j = 0, \pm 1, \ldots \right\}$

Anti aliasing filter

$t = 0, \pm h, \pm 2h, \ldots$

$$\left\{ \begin{matrix} \lambda + \frac{2\pi j}{h} \\ \lambda - \frac{2\pi j}{h} \end{matrix}, \ j = 0, \pm 1, \ldots \right\}$$

π/h Nyquist/folding frequency

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E.g. $X(t) = R\cos(\omega t + \varphi)$ $\varphi \sim U(-\pi, \pi)$

$$\gamma(h) = \frac{1}{2} R^2 \cos \omega h$$

continuous case

$$\frac{1}{4} R^2 \left[\delta(\lambda - \omega) + \delta(\lambda + \omega) \right]$$

discrete case

$$\frac{1}{4} R^2 \left[\eta(\lambda - \omega) + \eta(\lambda + \omega) \right]$$

E.g. <u>sample mean</u>

$$\bar{X}_n \quad \frac{1}{n}\sum_{t=0}^{n-1} X_t = \frac{1}{n}\int_{-\pi}^{\pi}\left(\sum_{t=0}^{n-1} e^{it\lambda}\right) dZ(\lambda)$$

$$\sum_{t=0}^{n-1} e^{it\lambda} = \Delta^n(\lambda)$$

$$\operatorname{var} \bar{X}_n = \frac{1}{n^2}\int_{-\pi}^{\pi} |\Delta^T(\lambda)|^2 f(\lambda)\,d\lambda$$

$$= \frac{1}{n^2}\int_{-\pi}^{\pi}\left(\frac{\sin n\lambda/2}{\sin \lambda/2}\right)^2 f(\lambda)\,d\lambda$$

$$\simeq \frac{2\pi}{n} f(0)$$

<u>DFT.</u>

$$A_k = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_t e^{-it\omega_k}$$