18 Feb 04

# Spectral approach to system id

$$y(t) = G(q)u(t) + v(t)$$
$$= \sum_k g(k)u(t-k) + v(t)$$
$$= s(t) + v(t)$$

$$\Phi_{yu}(\omega) = G(e^{i\omega})\Phi_u(\omega)$$

$$Y_N(\omega) = S_N(\omega) + V_N(\omega)$$

$$S_N(\omega) \approx G(e^{i\omega})U_N(\omega)$$

$\frac{2\pi k}{N} \approx \omega$ for $k_1 \le k \le k_2$

$G(\ )$ smooth, e.g. $\sum_k |g(k)| < \infty$

$$Y_N\left(\frac{2\pi k}{N}\right) \approx G\left(\exp\left\{i\frac{2\pi k}{N}\right\}\right)U_N\left(\frac{2\pi k}{N}\right) + V_N\left(\frac{2\pi k}{N}\right)$$
$$\approx G(e^{i\omega})U_N\left(\frac{2\pi k}{N}\right) + V_N\left(\frac{2\pi k}{N}\right)$$

18 Feb. 04

Suppose $v(t)$ stationary mixing

$$V_N\left(\frac{2\pi k}{N}\right) \sim IN^C(0, \Phi_N(\omega))$$

Set up correspondence with regression model

$$\{Y_j ; 1 \le j \le n\} \equiv \{Y_N\left(\frac{2\pi k}{N}\right) ; k_1 \le k \le k_2\}$$

$$a = G(e^{i\omega})$$

$$\{X_j ; 1 \le j \le n\} \equiv \{U_N\left(\frac{2\pi k}{N}\right) ; k_1 \le k \le k_2\}$$

$$\{\epsilon_j ; 1 \le j \le n\} \equiv \{V_N\left(\frac{2\pi k}{N}\right) ; k_1 \le k \le k_2\}$$

The $\epsilon_j$ will be approx $IN^C(0, \sigma^2)$ with

$$\sigma^2 = \Phi_N(\omega)$$

 18 Feb 04

## The estimates

$$\hat{G}_N(e^{i\omega}) = \hat{a} = \sum_j Y_j \bar{X}_j \Big/ \sum_j |X_j|^2$$

$$= \sum_k Y_N\left(\frac{2\pi k}{N}\right)\overline{U_N\left(\frac{2\pi k}{N}\right)} \Big/ \sum_k \left|U_N\left(\frac{2\pi k}{N}\right)\right|^2$$

$$= \hat{\Phi}^N_{yu}(\omega) \big/ \hat{\Phi}^N_u(\omega)$$

$$\hat{\Phi}^N_N(\omega) \approx \hat{\sigma}^2 = \sum_j |Y_j - \hat{a}X_j|^2/(n-1)$$

$$= \sum_k \left| Y_N\left(\frac{2\pi k}{N}\right) - \hat{G}_N(e^{i\omega})U_N\left(\frac{2\pi k}{N}\right)\right|^2$$

$$= \hat{\Phi}^N_y(\omega) - |\hat{\Phi}^N_{yu}(\omega)|^2 \big/ \hat{\Phi}^N_u(\omega)$$

$$= \hat{\Phi}^N_y(\omega)\left[1 - (\hat{K}^N_{yu}(\omega))^2\right]$$

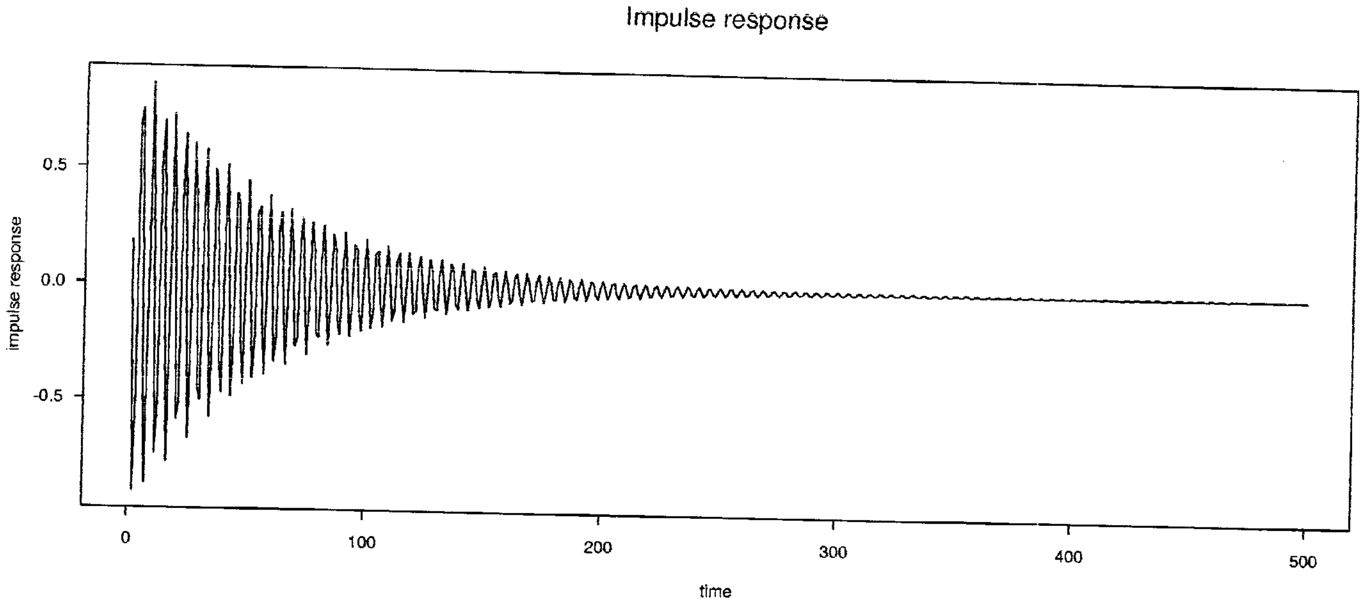
**Coherence**

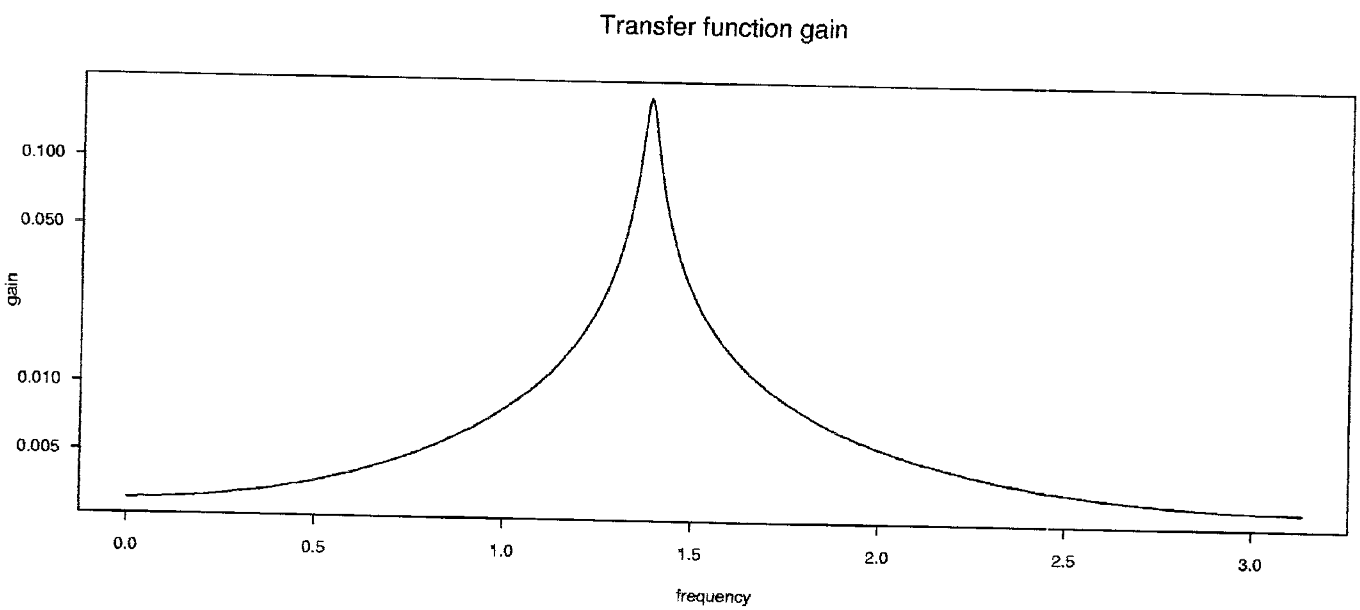
$$\left(K^N_{yu}(\omega)\right)^2 = |\hat{\Phi}^N_{yu}|^2 \big/ \hat{\Phi}^N_y(\omega)\hat{\Phi}^N_u(\omega)$$

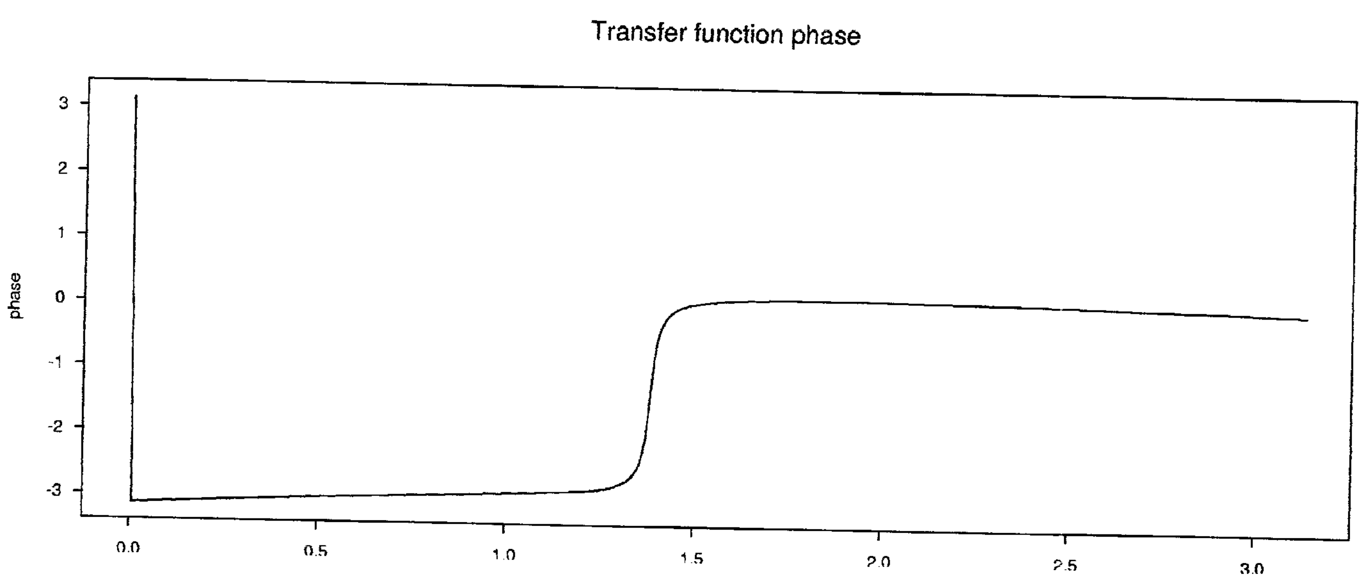
$$= \left|\sum_j Y_j \bar{X}_j\right|^2 \Big/ \sum_j |Y_j|^2 \sum_j |X_j|^2$$

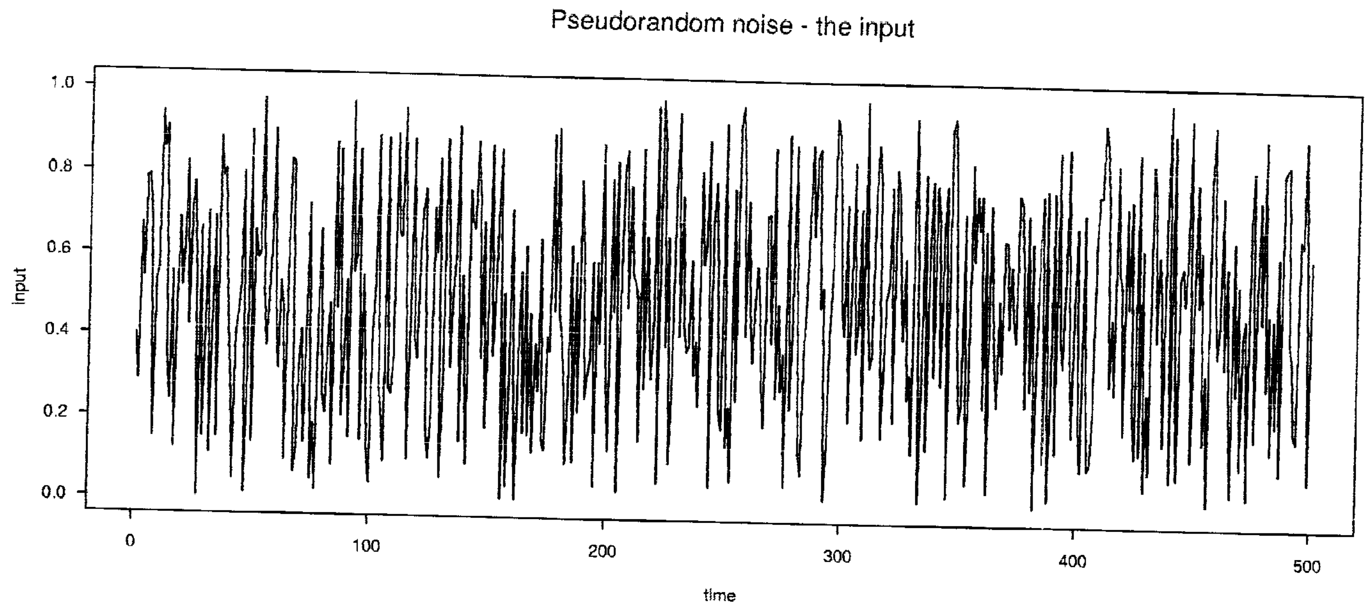
$$= 1 - \hat{\Phi}^N_N(\omega) \big/ \hat{\Phi}^N_y(\omega)$$

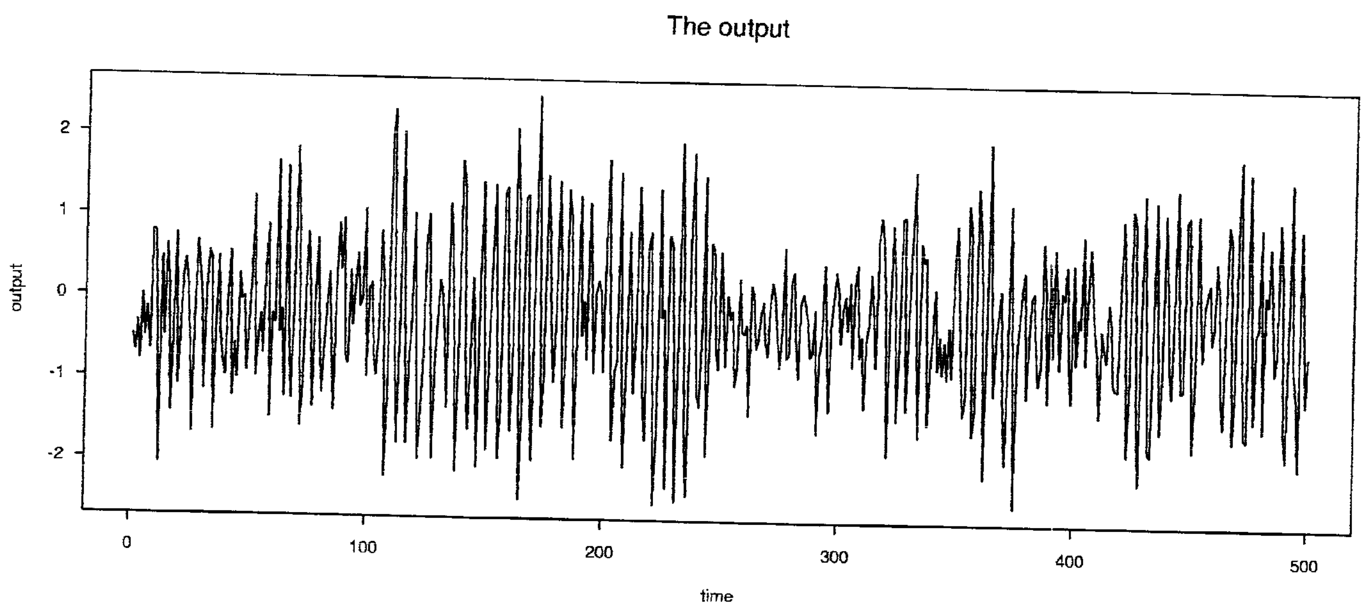
Terminology problem

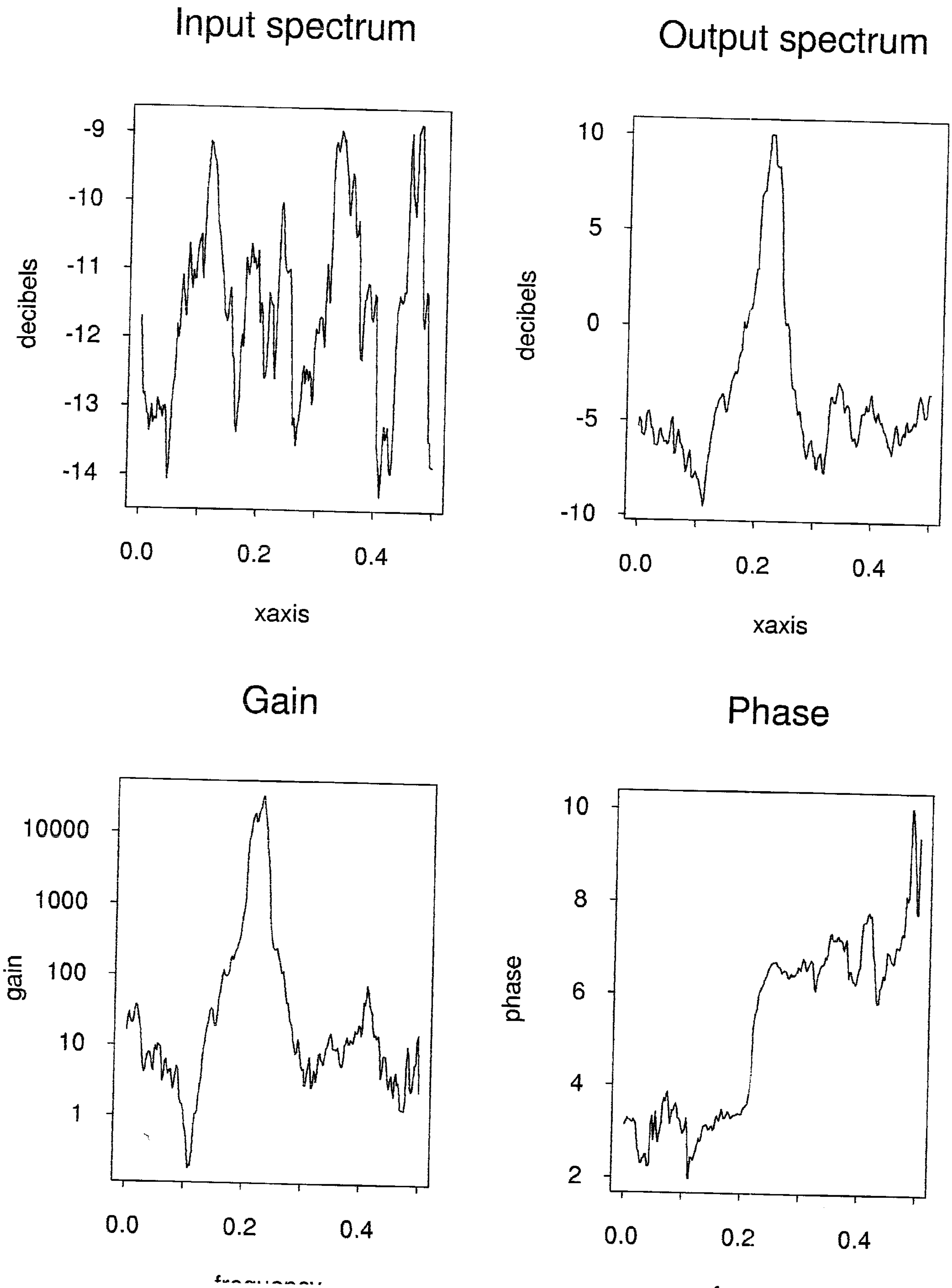

Impulse response
impulse response
0.5
0.0
-0.5
0
100
200
300
400
500
time

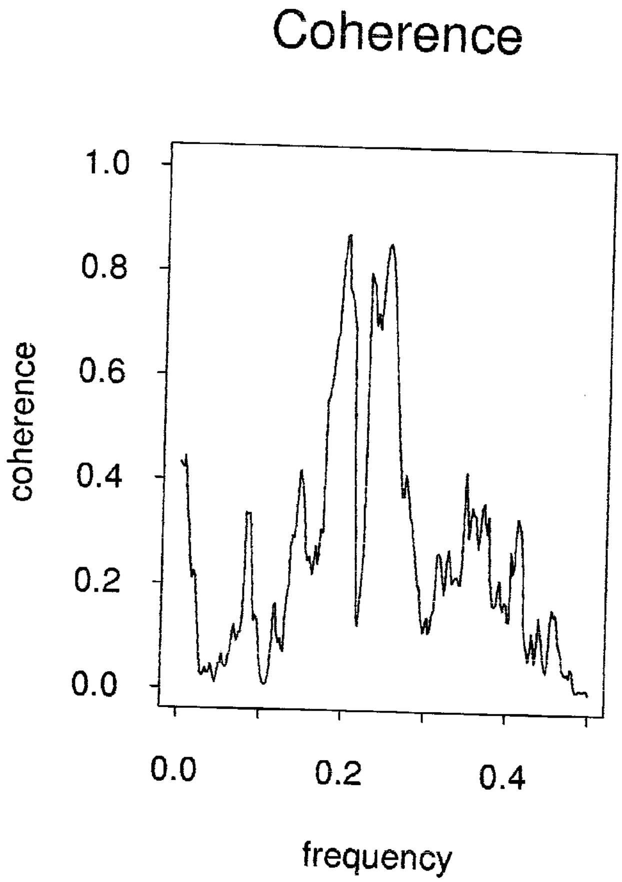

Transfer function gain
gain
0.100
0.050
0.010
0.005
0.0
0.5
1.0
1.5
2.0
2.5
3.0
frequency


Transfer function phase
phase
3
2
1
0
-1
-2
-3
0.0
0.5
1.0
1.5
2.0
2.5
3.0


Pseudorandom noise - the input
input
1.0
0.8
0.6
0.4
0.2
0.0
0
100
200
300
400
500
time


The output
output
2
1
0
-1
-2
0
100
200
300
400
500
time


Input spectrum
decibels
-9
-10
-11
-12
-13
-14
0.0
0.2
0.4
xaxis
Output spectrum
decibels
10
5
0
-5
-10
0.0
0.2
0.4
xaxis
Gain
gain
10000
1000
100
10
1
0.0
0.2
0.4
Phase
phase
10
8
6
4
2
0.0
0.2
0.4


Coherence
1.0
0.8
0.6
0.4
0.2
0.0
coherence
0.0
0.2
0.4
frequency

18 Feb. 04

The properties.

$$\hat{G}_N(e^{i\omega}) \sim N^c(G(e^{i\omega}), \Phi_v(\omega)/M\hat{\Phi}_u^N(\omega))$$

$\perp\!\!\!\perp$

$$\hat{\Phi}_v^N(\omega) \sim \Phi_v(\omega)\,\chi^2_{2(M-1)}/2(M-1)$$

18 Feb. 04

Notes.

1. Might take

$$\hat{\Phi}^N_{yu}(\omega) = \int_{-\pi}^{\pi} W_\gamma(\xi-\omega) Y_N(\xi) \overline{U_N(\xi)} \, d\xi$$

$$\hat{G}_N(e^{i\omega}) = \hat{\Phi}^N_{yu}(\omega) / \hat{\Phi}^N_u(\omega)$$

$$\hat{\Phi}^N_v(\omega) = \int_{-\pi}^{\pi} W_\gamma(\xi-\omega) \left| Y_N(\xi) - \hat{G}_N(e^{i\xi}) U_N(\xi) \right|^2 d\xi$$

18 Feb. 04

Coherence/coherency

$$(K_{yu}(\omega))^2 = |\Phi_{yu}(\omega)|^2 / \Phi_y(\omega)\Phi_u(\omega)$$

$$0 \leq (K_{yu}(\omega))^2 \leq 1$$

Clearly $(K_{yu}(\omega))^2 \geq 0$ (0 when $\Phi_{yu}(\omega) = 0$)

Consider

$$v(t) = y(t) - G(q)u(t)$$

with $G(e^{i\omega}) = \Phi_{yu}(\omega)/\Phi_u(\omega)$

$$\Phi_v(\omega) = \Phi_y(\omega) - G(e^{i\omega})\Phi_{uy}(\omega) - \overline{G(e^{i\omega})}\Phi_{yu}(\omega) + |G(e^{i\omega})|^2\Phi_u(\omega)$$

$$= \Phi_y(\omega) - |\Phi_{yu}(\omega)|^2 / \Phi_u(\omega)$$

$$= \Phi_y(\omega)\left[1 - (K_{yu}(\omega))^2\right]$$

so $(K_{yu}(\omega))^2 \leq 1$

$(K_{yu}(\omega))^2 = 1 \;\; \forall \omega$ iff $y(t) = G(q)u(t)$

$$|G(e^{i\omega})| = |\Phi_{yu}(\omega)|/\Phi_u(\omega) = \sqrt{|K_{yu}(\omega)|^2\Phi_y(\omega)/\Phi_u(\omega)}$$

18 Feb 04

Null distribution of $(\hat{R}_{yu}(\omega))^2$ has approx

expected value $1/M$

variance $1/M^2$

$100\alpha$ % point $1-(1-\alpha)^{1/(M-1)}$

18 Feb 04

<u>Effective degrees of freedom.</u>

$$\hat{\Phi}^N_N(\omega) = \int_{-\pi}^{\pi} W_\gamma(\xi-\omega)|V_N(\xi)|^2 d\xi$$

$$E\,\hat{\Phi}^N_N(\omega) \approx \Phi_N(\omega)$$

$$\text{var}\,\hat{\Phi}^N_N(\omega) \approx \Phi_N(\omega)^2 \int_{-\pi}^{\pi} W_\gamma(\xi)^2 d\xi\, \frac{2\pi}{N}$$

Approximate by $\Phi_N(\omega)^2 \chi^2_{2M}/2M$

Equate variances

$$\frac{1}{M} = \int_{-\pi}^{\pi} W_\gamma(\xi)^2 d\xi\, \frac{2\pi}{N}$$

$$M = N \Big/ 2\pi \int_{-\pi}^{\pi} W_\gamma(\xi)^2 d\xi$$

18 Feb 04

Splus

spec.pgram(x, plot, spans, taper, pad, detrend, demean)

spans = M

taper = 0

detrend = F

demean = T

plot = F

estimated spectrum in units of decibels

$10 \log_{10}$ (power)

R

help(spec.pgram, package = ts)