$v(t)$ zero mean, stationary, mixing

$$V_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} v(t) e^{-i\omega t}$$

I. $\sim N^c(0, \Phi_N(\omega))$ as $N \to \infty$
CLT

$\frac{2\pi k}{N} \approx \omega$ for $k_1 \le k \le k_2$

II. $V_N\left(\frac{2\pi k}{N}\right) \sim IN^c(0, \Phi_N(\omega))$

Periodogram.

$|V_N(\omega)|^2 \sim \Phi_N(\omega) \chi_2^2/2$ from I

$E|V_N(\omega)|^2 \approx \Phi_N(\omega)$

$\mathrm{var}|V_N(\omega)|^2 \approx \Phi_N(\omega)^2$ (*)

$$E|V_N(\omega)|^2 = \int_{-\pi}^{\pi} \frac{1}{2\pi N} \left[\frac{\sin N\xi/2}{\sin \xi/2}\right]^2 \Phi_N(\omega - \xi)\, d\xi$$

asymptotically unbiased
generally inconsistent

(*) suggests log as variance stabilizing transform

Estimation of the power spectrum.

From II

$$\left|V_N\left(\frac{2\pi k}{N}\right)\right|^2, \quad k_1 \le k \le k_2 \qquad \frac{2\pi k}{N} \sim \omega$$

$$\sim \text{indep } \Phi_N(\omega)\,\chi_2^2/2$$

Consider

$$\hat{\Phi}_N(\omega) = \sum_{k=k_1}^{k_2} \left|V_N\left(\frac{2\pi k}{N}\right)\right|^2 \Big/ \sum_{k=k_1}^{k_2} 1 \qquad (**)$$

$$\frac{2\pi k_1}{N} \geq 0$$

$$\sim \Phi_N(\omega)\,\chi_{2M}^2/2M$$

$$M = \sum_{k=k_1}^{k_2} 1$$

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$$E\,\hat{\Phi}_N(\omega) \approx \Phi_N(\omega)$$

$$\operatorname{var}\hat{\Phi}_N(\omega) \approx \Phi_N(\omega)^2/M$$

Want: N large (CLT)

$\frac{2\pi k}{N} \sim \omega$ (approx unbiased)

M large (consistent)

(**) is a <u>direct estimate</u>

Use fft( )

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Use of spectrum estimate (and periodogram).

Examination for white noise

data $v(t)$, $t = 1, \ldots, N$
output of Splus - runif( )

H_0: uncorrelated, $R_v(\tau) = 0$ for $\tau \neq 0$

$$\Phi_v(\omega) = \sum_\tau R_v(\tau) \exp\{i\omega\tau\}, \quad 0 \le \omega \le \pi$$

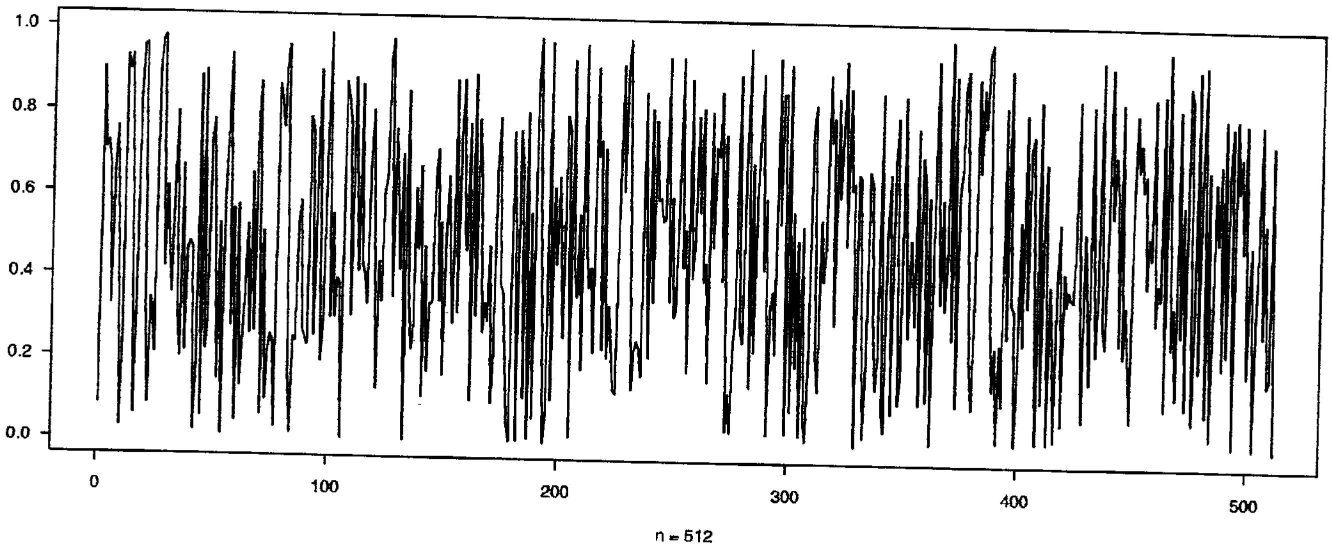
$$= \lambda \left(= \text{var } v(t)\right)$$

Really pseudo-random numbers

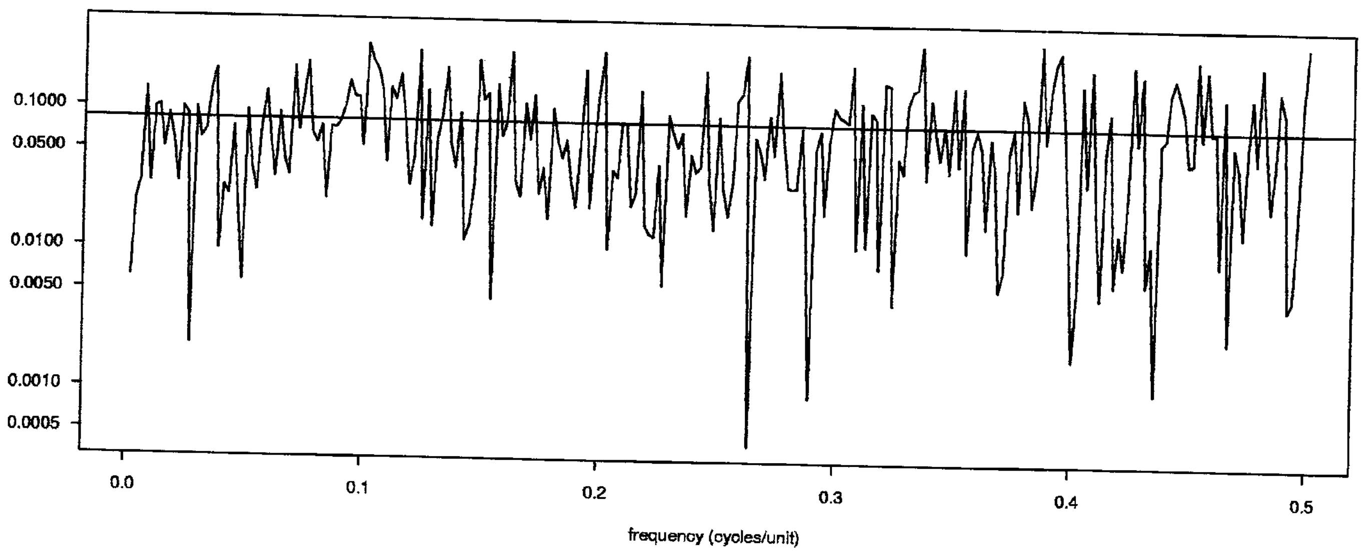
A 32-bit integer is produced by combining the outputs of two separate generators that proceed recursively.

E.g. $v_{t+1} = 4 v_t (1 - v_t)$ $\quad v_0 \in (0,1)$

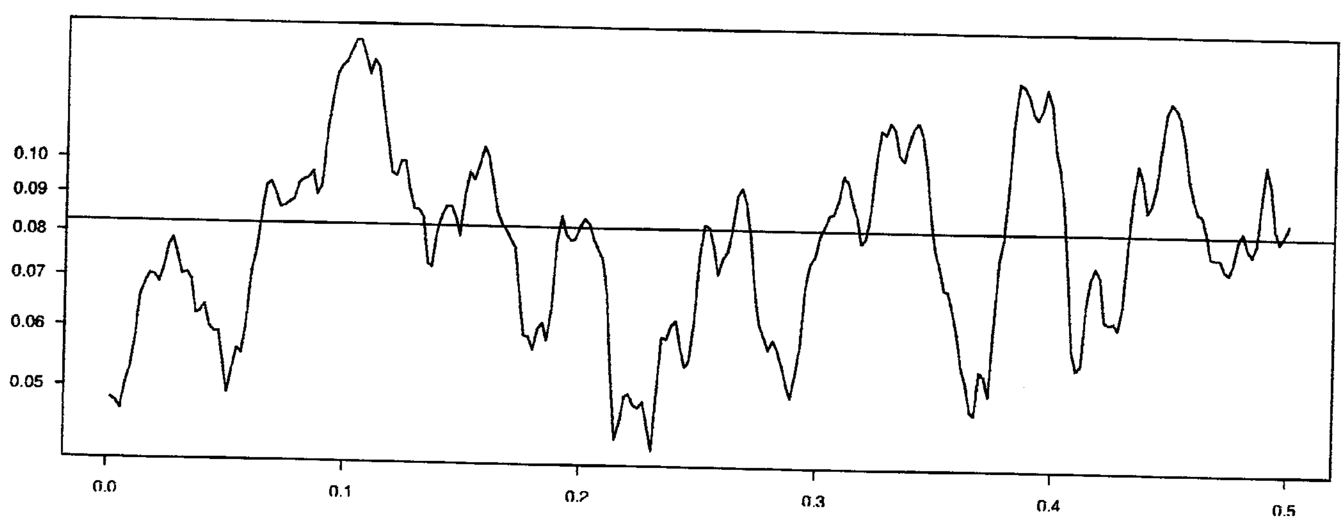
Pseudorandom from Spius



Periodogram



Smoothed periodogram, length 13



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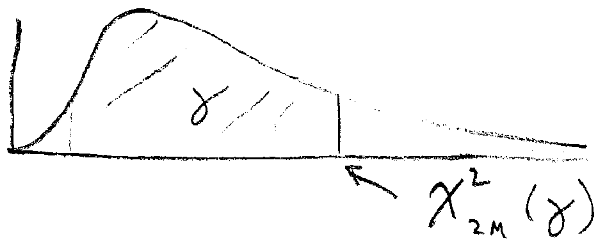
Approximate confidence interval.

$\hat{\Phi}_N(\omega)$ is approx $\Phi_N(\omega)\chi^2_{2M}/2M$



$$\text{Prob}\{\chi^2_{2M} < \chi^2_{2M}(\gamma)\} = \gamma$$

$$\text{Prob}\{\chi^2_{2M}(\tfrac{\alpha}{2}) < \chi^2_{2M} < \chi^2_{2M}(1-\tfrac{\alpha}{2})\} = 1-\alpha$$

$100\beta = 100(1-\alpha)\%$ CI

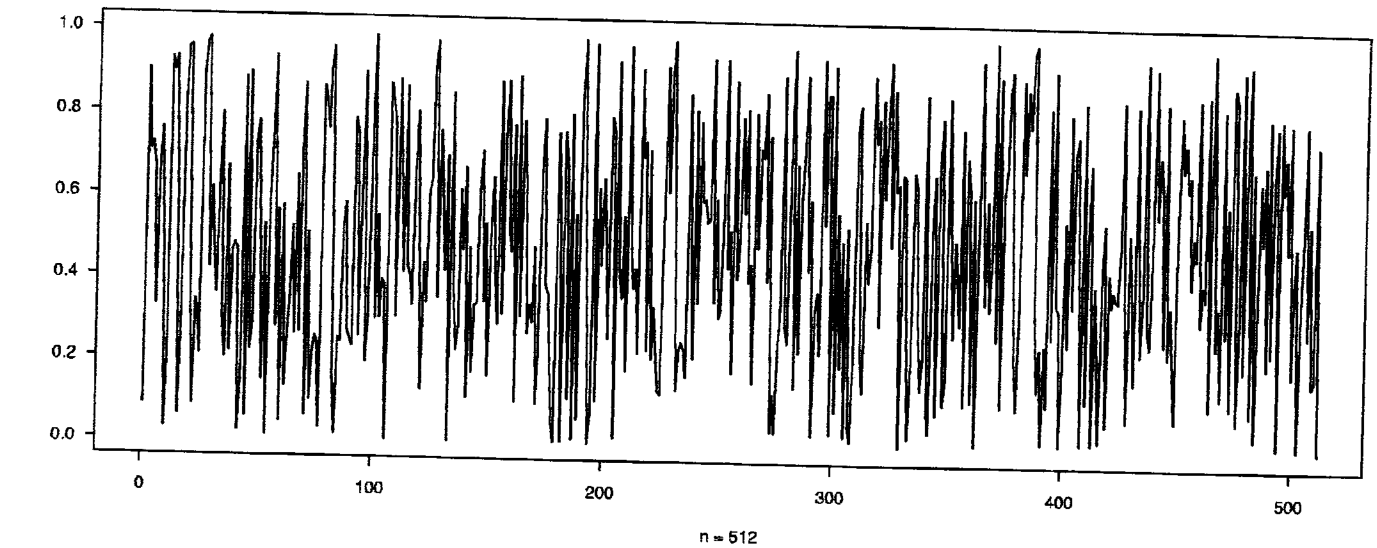
$$\text{Prob}\left\{\chi^2_{2M}(\tfrac{\alpha}{2}) < \frac{2M\hat{\Phi}_N(\omega)}{\Phi_N(\omega)} < \chi^2_{2M}(1-\tfrac{\alpha}{2})\right\} = 1-\alpha$$

$$\frac{2M\hat{\Phi}_N(\omega)}{\chi^2_{2M}(1-\frac{\alpha}{2})} < \Phi_N(\omega) < \frac{2M\hat{\Phi}_N(\omega)}{\chi^2_{2M}(\frac{\alpha}{2})}$$

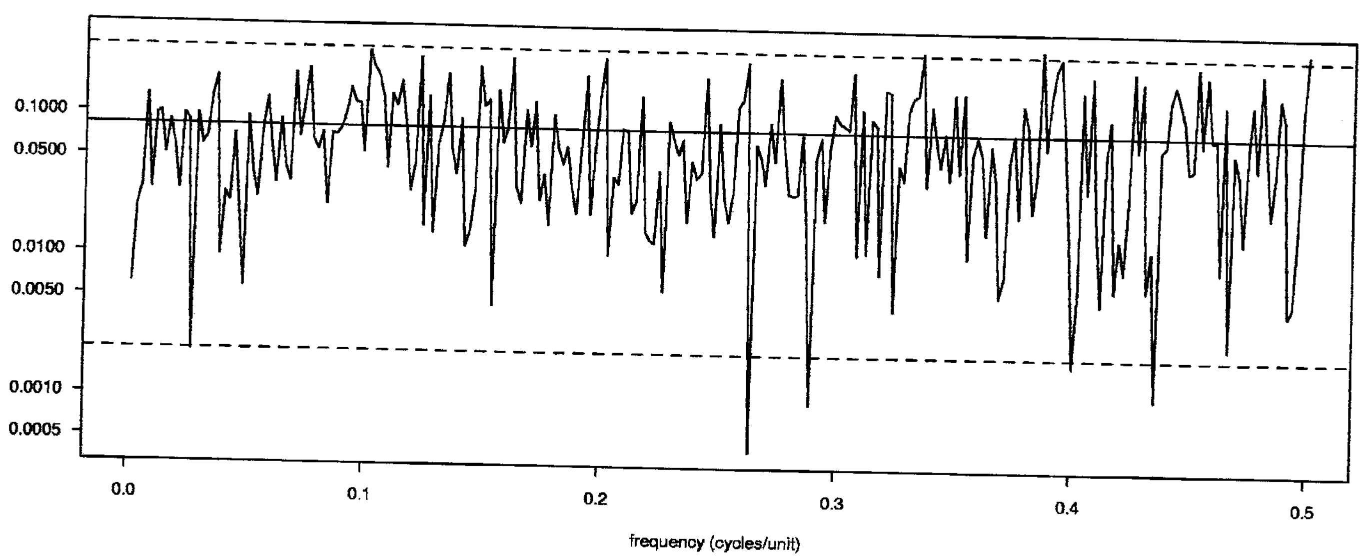
Take logs

qchisq( )

Pseudorandom from Splus



Periodogram



Smoothed periodogram, length 13

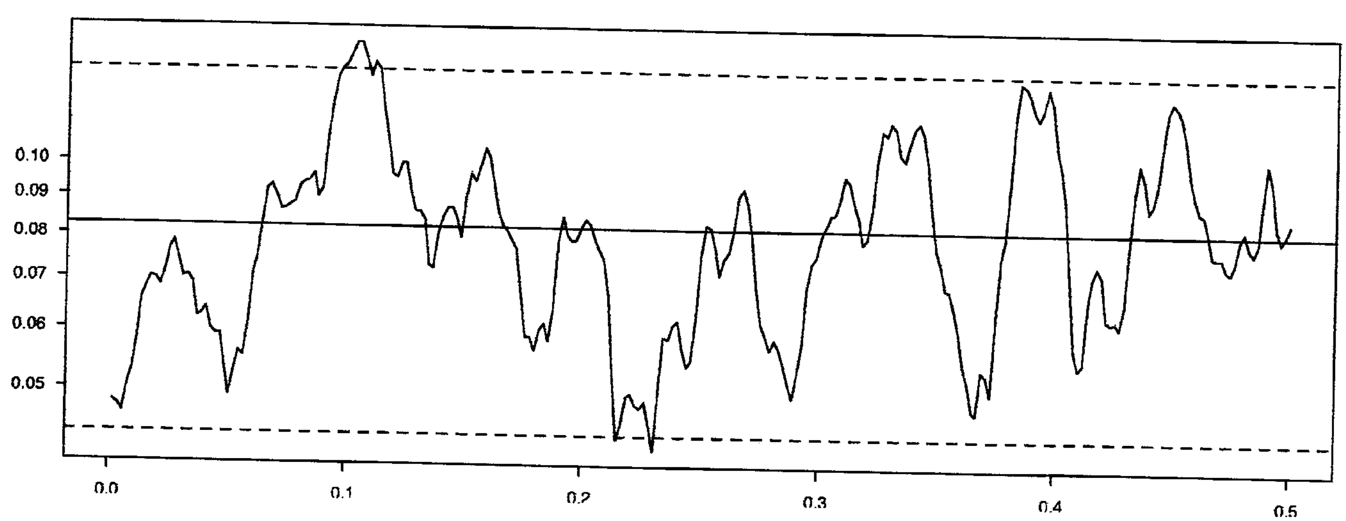


TABLE 6.1 Some Windows for Spectral Analysis

	$2\pi W_\gamma(\omega)$	$w_\gamma(\tau), \quad 0 \le \lvert\tau\rvert \le \gamma$
Bartlett	$\frac{1}{\gamma}\left(\frac{\sin \gamma\omega/2}{\sin \omega/2}\right)^2$	$1 - \frac{\lvert\tau\rvert}{\gamma}$
Parzen	$\frac{4(2+\cos\omega)}{\gamma^3}\left(\frac{\sin \gamma\omega/4}{\sin \omega/2}\right)^4$	$\begin{cases} 1 - \frac{6\tau^2}{\gamma^2}\left(1 - \frac{\lvert\tau\rvert}{\gamma}\right), & 0 \le \lvert\tau\rvert \le \frac{\gamma}{2} \\ 2\left(1 - \frac{\lvert\tau\rvert}{\gamma}\right)^3, & \frac{\gamma}{2} \le \lvert\tau\rvert \le \gamma \end{cases}$
Hamming	$\frac{1}{2}D_\gamma(\omega) + \frac{1}{4}D_\gamma(\omega - \pi/\gamma)$ $+\frac{1}{4}D_\gamma(\omega + \pi/\gamma)$, where $D_\gamma(\omega) = \frac{\sin(\gamma + \frac{1}{2})\omega}{\sin \omega/2}$	$\frac{1}{2}\left(1 + \cos\frac{\pi\tau}{\gamma}\right)$

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A class of direct estimates

$$\hat{\Phi}_N^N(\omega) = \int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\,|V_N(\xi)|^2\,d\xi$$

$W_\gamma(\omega)$: <u>frequency window</u>, approx delta γ large

$$\int_{-\pi}^{\pi} W_\gamma(\xi)\,d\xi = 1 \quad , \quad \int_{-\pi}^{\pi} \xi\, W_\gamma(\xi)\,d\xi = 0$$

$$w_\gamma(\tau) = \int_{-\pi}^{\pi} W_\gamma(\xi)\,e^{i\xi\tau}\,d\xi$$

<u>time window</u>

Table 6.1

$$E\,\hat{\Phi}_N^N(\omega) \approx \int_{-\pi}^{\pi} W_\gamma(\xi-\omega)\,\Phi_N(\xi)\,d\xi \qquad (***)$$

$$\text{var}\,\hat{\Phi}_N^N(\omega) \approx \Phi_N(\omega)^2 \int_{-\pi}^{\pi} W_\gamma(\xi)^2\,d\xi\;\frac{2\pi}{N} \qquad \left(\begin{smallmatrix}**\\**\end{smallmatrix}\right)$$

(**) suggests log for variance stabilization

(***) indicates that the estimate is generally biased

Note if $\Phi_v(\varepsilon)$ is constant, white noise

The bias of $\hat{\Phi}_v(\omega)$ may be reduced by <u>prefiltering</u>

$$v_F(t) = L_v(q)\, v(t)$$

with $L_v(q)$ chosen so that $v_F(t)$ has spectrum, $\Phi^F(\omega)$ approx constant

$$\Phi^F(\omega) = |L_v(e^{i\omega})|^2 \, \Phi_v(\omega)$$

may be inverted for ~~estimate~~ of $\Phi_v(\omega)$

E.g. $L_v(q) = 1 - \rho q$

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The indirect estimate.

$$\Phi_N(\omega) = \sum_\tau R_N(\tau)\, e^{-i\omega\tau}$$

Consider

$$\sum_\tau \hat{R}_N(\tau)\, e^{-i\omega\tau} \qquad \binom{***}{**}$$

with

$$\hat{R}_N(\tau) = \frac{1}{N} \sum_{1\le t,\, t-\tau\le N} v(t)\, v(t-\tau)$$

$$|V_N(\omega)|^2 = \frac{1}{N} \sum_t \sum_s v(t) v(s)\, e^{-i\omega t} e^{i\omega s}$$

$s = t - \tau$

$$= \frac{1}{N} \sum_{1\le t,\, t-\tau\le N} v(t)\, v(t-\tau)\, e^{-i\omega\tau}$$

$\binom{***}{**}$ is the periodogram!

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The indirect ~~estimate~~ is:

$$\sum_\tau w_\gamma(\tau) \hat{R}_v(\tau) e^{-i\omega\tau}$$

$w_\gamma(\tau)$ is called the <u>lag window</u>

Connections

$$w_\gamma(\tau) = \int_{-\pi}^{\pi} W_\gamma(\xi) e^{i\xi\tau} d\xi, \quad \tau = 0, \pm 1, \ldots$$

$$W_\gamma(\xi) = \frac{1}{2\pi} \sum_\tau w_\gamma(\tau) e^{-i\xi\tau}$$

$$w_\gamma(0) = 1$$

Problem 6T.1 $\quad w_\gamma(\tau) = w(\tau/\gamma)$

$w(0) = 1, \quad w(x) = 0$ for $|x| > 1$

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Notes.

1. Non-zero mean.
   Subtract sample mean

   ($\equiv$ avoiding $|V_N(0)|^2$ in smoothing)

2. Average periodograms for shorter stretches,
   e.g. $N = RM$

$$V_R^{(k)}(\omega) = \frac{1}{\sqrt{R}} \sum_{l=1}^{R} v((k-1)R + l) e^{-i\omega l}$$

$$\frac{1}{M} \sum_{k=1}^{M} |V_R^{(k)}(\omega)|^2$$

$\exists$ CLT etc.

3. Tapering 66.5 $\{h_t\}_1^N$, $\sum_1^N h_t^2 = 1$

$$V_N^T(\omega) = \sum_{t=1}^{N} h_t v(t) e^{-it\omega}$$

$$E|V_N^T(\omega)|^2 = \int_{-\pi}^{\pi} |H_N(\omega - \xi)|^2 \Phi_v(\xi) d\xi$$

$$H_N(\omega) = \sum_1^N h_t e^{-it\omega}$$

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Ordinary regression analysis (DRB book)

$$\underset{1\times n}{\underset{\sim}{Y}} = \underset{1\times k}{\underset{\sim}{a}}\,\underset{k\times n}{\underset{\sim}{X}} + \underset{1\times n}{\underset{\sim}{\epsilon}}$$

$\begin{bmatrix} \underset{\sim}{Y} \\ \underset{\sim}{X} \end{bmatrix}$ given

$E\underset{\sim}{\epsilon} = 0$, $E\underset{\sim}{\epsilon}^T \underset{\sim}{\epsilon} = \sigma^2 \underset{\sim}{I}$

unknowns $\underset{\sim}{a}, \sigma$.

$$\min_{\underset{\sim}{a}} \|\underset{\sim}{Y} - \underset{\sim}{a}\underset{\sim}{X}\|^2$$

given by $\hat{\underset{\sim}{a}} = \underset{\sim}{Y}\underset{\sim}{X}^T(\underset{\sim}{X}\underset{\sim}{X}^T)^{-1}$ if $\underset{\sim}{X}\underset{\sim}{X}^T$ nonsingular

$E\hat{\underset{\sim}{a}} = \underset{\sim}{a}$, $\operatorname{var}\hat{\underset{\sim}{a}} = \sigma^2(\underset{\sim}{X}\underset{\sim}{X}^T)^{-1}$

$$\hat{\sigma}^2 = \|\underset{\sim}{Y} - \hat{\underset{\sim}{a}}\underset{\sim}{X}\|^2$$

$$E\hat{\sigma}^2 = \sigma^2$$

If $\underset{\sim}{\epsilon}$ is $N_n(0, \sigma^2 \underset{\sim}{I})$

$$\hat{\underset{\sim}{a}}^T \text{ is } N_k(\underset{\sim}{a}^T, \sigma^2(\underset{\sim}{X}\underset{\sim}{X}^T)^{-1})$$

$$\underline{\|}$$

$$\hat{\sigma}^2 \sim \sigma^2 \chi^2_{n-k}/(n-k)$$

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Squared sample coefficient of multiple correlation

$$\hat{R}^2 = 1 - \|\underset{\sim}{Y} - \hat{a}\underset{\sim}{X}\|^2 / \|\underset{\sim}{Y}\|^2$$

E.g.

$$Y_j = aX_j + \epsilon_j$$

$$\hat{a} = \sum_j X_j Y_j / \sum_j X_j^2$$

$$\hat{\sigma}^2 = \sum_{j=1}^{n} (Y_j - \hat{a}X_j)^2 / (n-1)$$

$$\text{var}\,\hat{a} = \sigma^2 / \sum_{j=1}^{n} X_j^2$$

$$\hat{R}^2 = \left(\sum_{j=1}^{n} X_j Y_j\right)^2 / \sum_{j=1}^{n} X_j^2 \sum_{j=1}^{n} Y_j^2$$

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Complex case.

$$\underset{\sim}{Y} = \underset{\sim}{a}\,\underset{\sim}{X} + \underset{\sim}{\epsilon}$$

$$E\,\underset{\sim}{\epsilon}^T \underset{\sim}{\epsilon} = \underset{\sim}{0}\,, \quad E\,\overline{\underset{\sim}{\epsilon}}^T \underset{\sim}{\epsilon} = \sigma^2 I$$

$\begin{bmatrix} \underset{\sim}{Y} \\ \underset{\sim}{X} \end{bmatrix}$ has complex entries

$$\min_{\underset{\sim}{a}} \| \underset{\sim}{Y} - \underset{\sim}{a}\,\underset{\sim}{X} \| = \min_{\underset{\sim}{a}} (\underset{\sim}{Y} - \underset{\sim}{a}\,\underset{\sim}{X})\,(\overline{\underset{\sim}{Y} - \underset{\sim}{a}\,\underset{\sim}{X}})^T$$

$$\hat{\underset{\sim}{a}} = \underset{\sim}{Y}\,\overline{\underset{\sim}{X}}^T (\underset{\sim}{X}\,\overline{\underset{\sim}{X}}^T)^{-1}$$

$$E\,(\hat{\underset{\sim}{a}} - \underset{\sim}{a})^T (\hat{\underset{\sim}{a}} - \underset{\sim}{a}) = \underset{\sim}{0}$$

$$E\,(\hat{\underset{\sim}{a}} - \underset{\sim}{a})^T\, \overline{(\hat{\underset{\sim}{a}} - \underset{\sim}{a})} = \sigma^2 (\underset{\sim}{X}\,\overline{\underset{\sim}{X}}^T)^{-1}$$

$$\hat{\sigma}^2 = \| \underset{\sim}{Y} - \hat{\underset{\sim}{a}}\,\underset{\sim}{X} \|^2$$

$$\hat{\underset{\sim}{a}}^T \sim N_k^C\left(\underset{\sim}{a}^T, \sigma^2 (\underset{\sim}{X}\,\overline{\underset{\sim}{X}}^T)^{-1}\right)$$

$$\perp\!\!\!\perp$$

$$\hat{\sigma}^2 \sim \sigma^2 \chi^2_{2(n-k)} / 2(n-k)$$