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Properties of the DFT of a stationary mixing process

$$V_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} v(t) e^{-i\omega t}$$

$$E V_N(\omega) = 0 \quad \text{if} \quad E v(t) = 0$$

$$V_N(\omega + 2\pi) = V_N(\omega)$$

$$\overline{V_N(\omega)} = V_N(-\omega)$$

Periodogram $\qquad |V_N(\omega)|^2$

Consider $\quad E V_N(\omega) V_N(-\xi)$

$$= \frac{1}{N} \sum_{r=1}^{N} \sum_{s=1}^{N} E v(r) v(s) e^{-i\omega r} e^{i\xi s}$$

$$E v(r) v(s) = R_v(r-s)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(r-s)\lambda} \Phi_v(\lambda) d\tau$$

$$\Phi_v(\lambda) = \sum_{\tau=-\infty}^{\infty} R_v(\tau) \exp\{-i\tau\lambda\}$$

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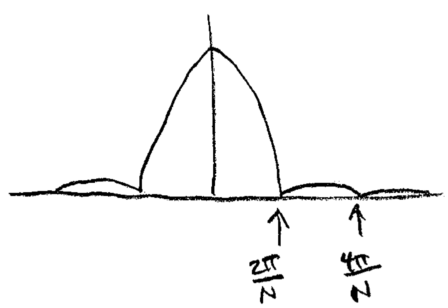
$$E\, V_N(\omega)\, V_N(-\xi)$$

$$= \frac{1}{2\pi N} \int_{-\pi}^{\pi} \sum_r \sum_s e^{-i\omega r} e^{i\xi s} e^{i(r-s)\lambda}\, \Phi_N(\lambda)\, d\lambda \qquad (*)$$

Define

$$\Delta^N(\omega) = \sum_{r=1}^{N} e^{-i\omega r} \qquad \text{(DRB book)}$$

$$|\Delta^N(\omega)| = |\sin N\omega/2| \,/\, |\sin \omega/2|$$



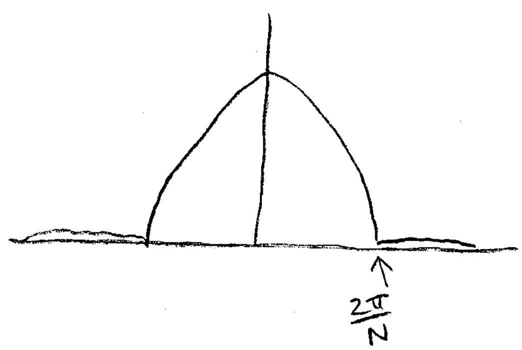
$$(*) = \frac{1}{2\pi N} \int_{-\pi}^{\pi} \Delta^T(\omega-\lambda)\, \overline{\Delta^T(\xi-\lambda)}\, \Phi_N(\lambda)\, d\lambda$$

$$E\,|V_N(\omega)|^2 = \frac{1}{2\pi N}\int_{-\pi}^{\pi} |\Delta^N(\omega-\lambda)|^2\, \Phi_v(\lambda)\, d\lambda$$

Féjer kernel

$$F_N(\omega) = \frac{1}{2\pi N}\left(\frac{\sin N\omega/2}{\sin \omega/2}\right)^2$$

$$F_N(\omega) \geq 0 \quad , \quad \int F_N(\omega)\, d\omega = 1$$



$$F_N(\omega) \longrightarrow \delta(\omega) \quad \text{as} \quad N \to \infty$$

$$E\,|V_N(\omega)|^2 \longrightarrow \Phi_v(\omega)$$ asymptotically unbiased

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$$E\, V_N(\omega) V_N(-\xi) \sim O(1/N) \qquad \omega \pm \xi \neq 0 \ (2\pi)$$

Under regularity conditions $V_N(\omega)$ is asymptotically normal.

It is (generally) complex-valued.

<u>Definition.</u> <u>Complex normal</u> with mean μ and variance σ^2

$$Z = U + iV$$

$U - \operatorname{Re}\mu$ and $V - \operatorname{Im}\mu$ are $IN(0, \sigma^2/2)$

<u>Properties.</u>

$$E\, Z = \mu$$
$$E\,|Z-\mu|^2 = \sigma^2$$

...

Theorems

1. $V_N(\omega)$ is asymptotically $N^C(0, \Phi_V(\omega))$
2. $V_N(\omega)$ and $V_N(\xi)$ with $\omega \pm \xi \neq 0 \ (2\pi)$ are asymptotically independent
3. $V_N\left(\frac{2\pi r}{N}\right)$ with $\frac{2\pi r}{N} \to \lambda$ is asymptotically $N^C(0, \Phi_V(\omega))$
4. $V_N\left(\frac{2\pi r}{N}\right), V_N\left(\frac{2\pi s}{N}\right)$ with $\frac{2\pi r}{N} \pm \frac{2\pi s}{N} \neq 0 \ (2\pi)$ are asymptotically independent.

Proof. See DRB book

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Chi-squared distribution, χ^2_f

$\chi^2_f = z_1^2 + \dots + z_f^2$ with z's $I\,N(0,1)$

$\chi^2_2/2$ is exponential mean 1

$E\,\chi^2_f = f$

$\mathrm{var}\,\chi^2_f = 2f$

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Large sample distribution of periodogram

1. $V_N(\omega) \sim N^c(0, \Phi_N(\omega))$

$\text{Re}\, V_N(\omega), \text{Im}\, V_N(\omega) \sim IN(0, \frac{1}{2}\Phi_N(\omega))$

$$|V_N(\omega)|^2 = (\text{Re}\, V_N(\omega))^2 + (\text{Im}\, V_N(\omega))^2$$

$$\sim \Phi_N(\omega)\, \chi^2_2/2$$

i.e. exponential

2. $|V_N(\omega)|^2$ and $|V_N(\xi)|^2$ with $\omega \pm \xi \neq 0 \ (2\pi)$

are asymptotically independent

3. $|V_N(\frac{2\pi r}{N})|^2$ and $|V_N(\frac{2\pi s}{N})|^2$ with $\frac{2\pi r}{N} \pm \frac{2\pi s}{N} \neq 0 \ (2\pi)$

are asymptotically independent

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Discussion.

The periodogram is an inconsistent estimate of $\Phi_v(\omega)$, unless $\Phi_v(\omega) = 0$.

$$|V_N(\omega)|^2 \xrightarrow{p} \Phi_v(\omega)\,\chi_2^2/2 \quad , \text{ a r.v.}$$

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Example.

$$y(t) = \alpha\cos(\beta t+\gamma) + v(t)$$
$$= s(t) + v(t)$$

$s(\cdot)$: signal

$$Y_N(\omega) = S_N(\omega) + V_N(\omega)$$

$$S_N(\omega) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} \alpha\cos(\beta t+\gamma)e^{-i\omega t}$$
$$= \frac{1}{\sqrt{N}}\sum_{t=1}^{N}\frac{\alpha}{2}e^{i\beta t+\gamma}e^{-i\omega t} + \frac{1}{\sqrt{N}}\sum_{t=1}^{N}\frac{\alpha}{2}e^{-i\beta t-\gamma-i\omega t}$$
$$= \frac{\alpha}{2\sqrt{N}}e^{i\gamma}\Delta^N(\omega-\beta) + \frac{\alpha}{\sqrt{N}}e^{-i\gamma}\Delta^N(\omega+\beta)$$

$$|Y_N(\omega)| \simeq$$

$$\Delta^N(0) = N$$