Chapter 2 continued

For $MA(1)$

$$w_{hh} \;=\; 1 \;-\; 3\rho(1)^2 \;+\; 4\rho(1)^4, \qquad h=1$$
$$=\; 1 \;+\; 2\rho(1)^2, \qquad h>1$$

For $AR(1)$

$$w_{hh} \;=\; (1 \;-\; \phi^{2h})(1 \;+\; \phi^2)(1 \;-\; \phi^2)^{-1} \;-\; 2h\phi^{2h}$$

Forecasting stationary time series

Predicting second-order r.v's. Wish to predict Y given W. Look for minimum

$$MSE \;=\; S(a_0,a_1) \;=\; E\{(Y-a_0-a_1W)^2\}$$

$$\mu_Y \;=\; a_0 + a_1\;\mu_W, \qquad cov\{W,W\}a_1 \;=\; \Gamma a \;=\; \gamma \;=\; cov\{Y,W\}$$

Predicting X_{n+h} from X_n

$$\gamma(0)a \;=\; \gamma(h)$$

AR(1). $Y_t-\mu=\phi(Y_{t-1}-\mu)+Z_t, |\phi|<1$. Predictor of Y_{n+1} given Y_n.

$$\mu+\phi(Y_n-\mu) \tag{1}$$

Vector case. Wish to predict Y given $\mathbf{W}$ using $a_0 \;+\; \mathbf{aW}$

$$\mu_Y \;=\; a_0 \;+\; \mathbf{a}\;\boldsymbol{\mu}_{\mathbf{W}}, \qquad \mathbf{\Gamma a} \;=\; \boldsymbol{\gamma} \tag{2}$$

$$\mathbf{\Gamma} \;=\; cov\{\mathbf{W},\mathbf{W}\}, \qquad \boldsymbol{\gamma} \;=\; cov\{Y,\mathbf{W}\}$$

Best linear predictor. Stationary case

$$P_nX_{n+h} \;=\; a_0+a_1X_n+...+a_nX_1 \tag{3}$$

$$MSE \;=\; S(a_0,...,a_n) \;=\; E(X_{n+h}-a_0-a_1X_n-...-a_nX_1)^2 \tag{4}$$

$$\mu \; = \; a_0 + a_1\mu + ... + a_n\mu \tag{5}$$

$$cov(X_{n+h}, X_{n+1-j}) \; = \; a_1 cov(X_n, X_{n+1-j}) + ... + a_n cov(X_1, X_{n+1-j}) \quad j = 1, ..., n$$

$$a_0 \; = \; \mu(1 - \sum_{i=1}^{n} a_i), \qquad \mathbf{\Gamma}_n \mathbf{a}_n \; = \; \boldsymbol{\gamma}_n(h) \tag{6}$$

The predictor

$$P_n X_{n+h} \; = \; \mu \; + \; \sum_{i=1}^{n} a_i(X_{n+1-i} - \mu) \tag{7}$$

The minimum MSE

$$E(X_{n+h} - P_n X_{n+h})^2$$

Properties of P_n.

4. $E[(X_{n+h} \; - \; P_n X_{n+h})X_j] \; = \; 0, \; j = 0, 1, ..., n$

or

$$E[(Error) \; \times \; (Predictor\ Variable)] \; = \; 0$$

This uniquely determines the predictor.
(The residuals are orthogonal to the regressors.)

$h = 1$ case of AR(1). $X_t \; = \; \phi X_{t-1} \; + \; Z_t$

$$P_n X_{n+1} \; = \; \phi X_n \qquad E(X_{n+1} \; - \; P_n X_{n+1})^2 \; = \; \sigma^2$$

Case of AR(p). $n \; > \; p$

$$P_n X_{n+1} \; = \; \phi X_n \; + \; ... \; + \; \phi_p X_{n+1-p} \qquad E(X_{n+1} \; - \; P_n X_{n+1})^2 \; = \; \sigma^2$$

Prediction using infinite past. $P_{n,m} X_{n+h}$ based on $1, \; X_m, \; ..., \; X_0, \; m \; < \; 0$

$$\tilde{P}_n \; = \; \lim_{m \to \infty} P_{m,n} X_{n+h} \; = \; \sum_{j=1}^{\infty} \alpha_j X_{n+1-j}$$

$$E[(X_{n+h} \; - \; \tilde{P}_n X_{n+h})X_{n+1-i}] \; = \; 0, \quad i = 1, 2, ...$$

Innovations. One-step predictors. Zero-mean case.

$$\hat{X}_1 \; = \; 0$$

$$\hat{X}_n \; = \; P_{n-1} X_n, \quad n = 2, 3, ... \qquad v_n \; = \; E(X_{n+1} \; - \; P_n X_{n+1})^2$$