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CLT's for FT's

Notation of DRB book & papers

$$d^T(\lambda) = \sum_{t=0}^{T-1} X(t) e^{-i\lambda t}, \quad -\infty < \lambda < \infty$$

Suppose $\{X(t)\}$ zero mean, stationary, mixing (defined later).

Results. As $T \to \infty$

1. $d^T(\lambda) \sim N^C(0, 2\pi T f(\lambda))$

   $\lambda \neq 0, \pm 2\pi, \ldots$

2. $d^T(\lambda), d^T(\mu)$ asymp independent

   λ, μ distinct

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Split data

3. $T = LV \qquad V \to \infty$

$$d^V(\lambda, l) = \sum_{j=0}^{V-1} e^{-i\lambda(j+(l-1)V)} X(j+(l-1)V)$$

$$d^V(\lambda, l) \sim N^C(0, 2\pi V f(\lambda)) \qquad \lambda \neq 0$$

$d^V(\lambda, l), d^V(\lambda, l')$ asymp independent

4. Case of $\frac{2\pi s}{T}$, s integer near λ

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Proofs.

I. If $\{X(t)\}$ Gaussian, normality is obvious.

II.

$\mathrm{cov}\{d^T(\lambda), d^T(\mu)\}$

$$= \sum_{s=0}^{T-1} \sum_{t=0}^{T-1} e^{-i\lambda s} e^{i\mu t} c_2(s-t)$$

$s-t=u$

$$= \sum_{u=-T+1}^{T-1} \sum_{\substack{0 \le t+u \le T-1 \\ 0 \le t \le T-1}} e^{-i\lambda(t+u)} e^{i\mu t} c_2(u)$$

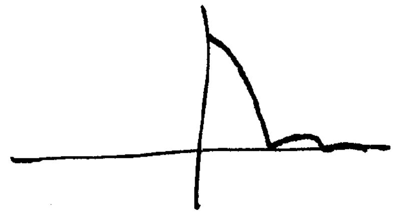
$$= \sum_{u=-T+1}^{T-1} e^{-i\lambda u} c_2(u) \left\{ \sum_{\max(0,-u) \le t \le \min(T-1, T-1-u)} e^{-i(\lambda-\mu)t} \right\}$$

Term in $\{ \quad \}$

$$\sim \sum_{t=0}^{T-1} e^{-i(\lambda-\mu)t} = \Delta^T(\lambda-\mu)$$

9 April 2003

$$|\Delta^T(\lambda)| = \left|\frac{\sin T\lambda/2}{\sin \lambda/2}\right|$$



bounded $\lambda \neq 0, \pm 2\pi, \ldots$

$$\Delta^T(0) = T$$

$$\operatorname{cov}\{d^T(\lambda), d^T(\mu)\} \sim 2\pi T f(\lambda) \qquad \lambda = \mu, \ldots$$

$$\sim O(1)$$

$$\lambda \pm \mu \neq 0, \pm 2\pi, \ldots$$

gives independence in Gaussian case

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Cumulants. Moments existing

Order 1. EX

Order 2. $\text{cov}\{X, Y\}$

Order k. Coefficient of $\theta_1 \ldots \theta_k$ in Taylor expansion of

$$\log E\{e^{i(\theta_1 X_1 + \ldots + \theta_k X_k)}\}$$

$$\text{cum}\{X_1, \ldots, X_k\}$$

Properties.

a) Multilinear

b) Vanish if some subset of X's independent of remainder

$\equiv$ measure of dependence

c) $\text{cum}_k = 0$ for normal and $k = 3, \ldots$

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Cumulant function of order k, stationary case

$$c_k(u_1, \ldots, u_{k-1})$$

$$= \mathrm{cum}\{X(t+u_1), \ldots, X(t+u_{k-1}), X(t)\}$$

Mixing

$$\sum_{u_1} \cdots \sum_{u_{k-1}} |c_k(u_1, \ldots, u_{k-1})| < \infty$$

7   9 April 2003

Proving a CLT when moments exist

$$S_T = X_0 + \dots + X_{T-1}$$

$$E S_T = T\mu$$

$$\mathrm{Var}\, S_T = T\sigma^2$$

$$\mathrm{cum}_k(S_T) = T K_k \qquad k = 3, 4, \dots$$

$$Z_T = \frac{S_T - T\mu}{\sqrt{T}\,\sigma}$$

$$E Z_T = 0$$

$$\mathrm{Var}\, Z_T = 1$$

$$\mathrm{cum}_k(Z_T) = \frac{K_k}{T^{\frac{k}{2}-1}\sigma^k} \to 0 \qquad k > 2$$

Normal is determined by its moments.

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Proving a CLT for a time series

$$S_T = X_0 + \dots + X_{T-1}$$

$$ES_T = T\mu$$

$$\operatorname{var} S_T = \sum_{s,t=0}^{T-1} c_2(s-t)$$

$$= T \sum_{u=-T+1}^{T-1} \left(1 - \frac{|u|}{T}\right) c_2(u)$$

$$\sim T \sum_{u=-\infty}^{\infty} c_2(u)$$

OK as $\sum_{u=-\infty}^{\infty} |c_2(u)| < \infty$

$$\operatorname{cum}_k(S_T) = \sum_{t_1} \dots \sum_{t_k} c_k(t_1 - t_k, \dots, t_{k-1} - t_k)$$

$$=$$

$$= O(T)$$

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Cumulant spectra

$$f_k(\lambda_1, \ldots, \lambda_{k-1})$$

$$= \frac{1}{(2\pi)^{k-1}} \sum_{u_1} \cdots \sum_{u_{k-1}} e^{-i(\lambda_1 u_1 + \cdots + \lambda_{k-1} u_{k-1})}$$

$$c_k(u_1, \ldots, u_{k-1})$$

$$-\infty < \lambda_1, \ldots, \lambda_{k-1} < \infty .$$