16 April 03

Regression in the complex case.

$$Y_j = \beta x_j + \epsilon_j \quad , \quad j = 1, \ldots, n$$

x's fixed

$$\epsilon_j : \; IN^C(0, \sigma^2)$$

$$EY_j = \beta x_j \qquad \text{var } Y_j = \sigma_j^2$$

LSE

$$\hat{\beta} = \sum_j{}' Y_j \bar{x}_j \Big/ \sum_j{}' |x_j|^2$$

$$= \beta + \sum_j{}' \epsilon_j \bar{x}_j \Big/ \sum_j |x_j|^2$$

$$\hat{\beta} \sim N^C(\beta, \sigma^2 / \sum_j |x_j|^2)$$

$$\hat{\sigma}^2 = \sum_j{}' |Y_j - \hat{\beta} x_j|^2 / (n-1)$$

$$\sim \sigma^2 \chi^2_{2(n-1)} / 2(n-1)$$

$$\hat{\beta} \perp\!\!\!\perp \hat{\sigma}^2$$

$$1 - |\hat{R}|^2 = \sum_j |Y_j - \hat{\beta} x_j|^2 \Big/ \sum_j |Y_j|^2$$

$$0 \le |\hat{R}|^2 \le 1$$

16 April 2003

Cross-spectral analysis

Separate regression at each frequency

Model

$$Y(t) = \mu + \sum_u{}' a(u) X(t-u) + \epsilon(t) \qquad (*)$$

$\{\epsilon(\cdot)\}$ 0-mean, stationary

$$EY(t) = \mu + \sum_u{}' a(u) X(t-u)$$

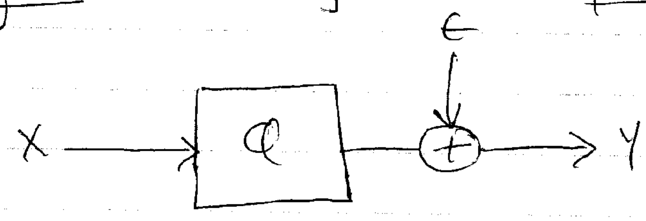
nonstationary

$\{a(\cdot)\}$ impulse response

$$A(\lambda) = \sum_u{}' e^{-i\lambda u} a(u)$$

transfer function

$|A(\lambda)|$: gain $\quad$ $\arg A(\lambda)$: phase



System identification

Estimate $\{a(\cdot)\}$, $\{A(\cdot)\}$, $f_{\epsilon\epsilon}(\lambda)$ from data $(X(t), Y(t))$, $t = 0, \dots, T-1$

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Pick n distinct $\frac{2\pi j}{T}$ near λ

FT (*)

$$d_Y^T(\lambda) \simeq A(\lambda) d_X^T(\lambda) + d_\epsilon^T(\lambda)$$

$$d_Y^T\left(\frac{2\pi j}{T}\right) \simeq A(\lambda) d_X^T\left(\frac{2\pi j}{T}\right) + d_\epsilon^T\left(\frac{2\pi j}{T}\right)$$

cp. $Y_j = \beta x_j + \epsilon_j$

CLT for $d_\epsilon^T\left(\frac{2\pi j}{T}\right)$: $IN^C(0, 2\pi T f_{\epsilon\epsilon}(\lambda))$

$Y_j \sim d_Y^T\left(\frac{2\pi j}{T}\right)$; $x_j \sim d_X^T\left(\frac{2\pi j}{T}\right)$; $\epsilon_j \sim d_\epsilon^T\left(\frac{2\pi j}{T}\right)$

$\beta \sim A(\lambda)$; $\sigma^2 \sim 2\pi T f_{\epsilon\epsilon}(\lambda)$

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$$A^T(\lambda) = \sum_j d_Y^T\left(\frac{2\pi j}{T}\right)\overline{d_X^T\left(\frac{2\pi j}{T}\right)} \Big/ \sum_j \left|d_X^T\left(\frac{2\pi j}{T}\right)\right|^2$$

$$\sim N^C\left(A(\lambda),\ 2\pi T f_{\epsilon\epsilon}(\lambda) \Big/ \sum_j \left|d_X^T\left(\frac{2\pi j}{T}\right)\right|^2\right)$$

$$\perp\!\!\!\perp$$

$$f_{\epsilon\epsilon}^T(\lambda) = \sum_j \left|d_Y^T\left(\frac{2\pi j}{T}\right) - A^T(\lambda) d_X^T\left(\frac{2\pi j}{T}\right)\right|^2 \Big/ 2\pi T(n-1)$$

$$\sim f_{\epsilon\epsilon}(\lambda)\ \chi^2_{2(n-1)} / 2(n-1)$$

$$\sum_j \left|d_X^T\left(\frac{2\pi j}{T}\right)\right|^2 \sim 2\pi T n f_{XX}^T(\lambda)$$

$$|R_{YX}^T(\lambda)|^2 = 1 - f_{\hat\epsilon\hat\epsilon}^T(\lambda) / f_{YY}^T(\lambda)$$

$$= |f_{YX}^T(\lambda)|^2 / f_{XX}^T(\lambda) f_{YY}^T(\lambda)$$

$$E|R_{YX}^T(\lambda)|^2 \sim 1/(n-1)$$

$100\alpha\%$ pt. $\quad 1 - (1-\alpha)^{1/(n-1)}$

$$f_{\epsilon\epsilon}^T(\lambda) = [1 - |R_{YX}^T(\lambda)|^2] f_{YY}^T(\lambda)$$

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$$f^T_{YX}(\lambda) = \frac{1}{2\pi T}\frac{1}{m}\sum_j d^T_Y\left(\frac{2\pi j}{T}\right)\overline{d^T_X\left(\frac{2\pi j}{T}\right)}$$

<u>Plot</u> $|R^T_{YX}(\lambda)|^2$

$\log|A^T(\lambda)|$

$\arg A^T(\lambda)$

$$AV(\log|A^T(\lambda)|) \sim \frac{1}{2m}\left[|R^T_{YX}(\lambda)|^{-2} - 1\right]$$

See DRB book Chapter 6

m can be $m_T \to \infty$, $\frac{m_T}{T} \to 0$

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Splitting approach

Suppose $T = LV$ having in mind

$V \to \infty$, L fixed

Let $j = 1, \ldots, L$

Compute

$$y_j = d_y^V(\lambda, j) = \sum_{v=0}^{V-1} e^{-i\lambda v} Y(v + \overline{j-1}\,V)$$

$$x_j = d_x^V(\lambda, j) =$$

Better for higher-order spectra

L can be L_T

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Importance of prefiltering

$$Y(t) = \alpha X(t-v) + \epsilon(t)$$

$$E f_{YX}^T(\lambda) \simeq \frac{1}{2\pi T} \frac{1}{m} \sum_j A\left(\frac{2\pi j}{T}\right) \left|d_X^T\left(\frac{2\pi j}{T}\right)\right|^2$$

$$\simeq \frac{1}{2\pi T} \frac{1}{m} \sum_j e^{i\frac{2\pi j v}{T}} \left|d_X^T\left(\frac{2\pi j}{T}\right)\right|^2$$

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Estimating impulse response

$$a(u) = \frac{1}{2\pi} \int_0^{2\pi} A(\lambda)\, e^{iu\lambda}\, d\lambda$$

$$\simeq \frac{1}{P} \sum_{p=0}^{P-1} A\left(\frac{2\pi p}{P}\right) e^{i 2\pi p u / P}$$

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System identification

Linear time invariant case

Forget noise for the moment

$$Y(t) = \sum_u a(t-u) X(u)$$

After $\{a(.)\}$, $\{A(.)\}$

I. Deterministic input

1. Pulse probe

$$X(t) = \delta\{t\} \implies Y(t) = a(t)$$

2. Periodic pulses

Suppose $a(u) = 0$, $u < 0$ or $u \geq U$

$$X(t) = \sum_{j=1}^{J} \delta\{t - j\Delta\} \quad \text{with } \Delta > U$$

Stack and average curves

Evoked response experiments

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3. Sinusoid.

$$X(t) = e^{i\lambda t}$$

$$Y(t) = \sum_u a(u) e^{i\lambda(t-u)}$$

$$= e^{i\lambda t} A(\lambda)$$

4. m-sequence (pseudorandom binary)

$X(t) = \pm 1$, period $N = 2^m - 1$

mean $1/N$

autcor $-1/N - 1/N^2$ ~ white

$$Y(t) = \sum_u a(u) X(t-u)$$

$$\sum_v Y(t) X(t-v) = \sum_u a(u) \sum_v X(t-u) X(t-v)$$

$$\sim V a(v)$$

e.g. $X_t = X_{t-1} X_{t-4} X_{t-6} X_{t-12}$ period 4095

$X_t = -1 \quad t = 1, \ldots, 12$

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5. General $\{X(t)\}$

$$c^T_{YX}(v) \sim \sum_u a(u)\, c^T_{XX}(v-u)$$

deconvolve

FT

$$f^T_{YX}(\lambda) \sim A(\lambda)\, f^T_{XX}(\lambda)$$

6. Chirps

$$A(\lambda) = 0 \qquad 0 \le \lambda < \lambda_0 \text{ and } \lambda_1 < \lambda$$

$$X(t) = \cos(\lambda_t t) \qquad 0 \le t \le T$$

$$\lambda_t = \lambda_0 + (\lambda_1 - \lambda_0)\frac{t}{T}$$

$$f^T_{XX}(\lambda) \sim \begin{cases} 0 & 0 \le \lambda < \lambda_0 \text{ and } \lambda_1 < \lambda \\ C & \lambda_0 \le \lambda \le \lambda_1 \end{cases}$$

Repeat, stack, average

sonar, radar, bats, VIBROSEIS

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II. Stochastic input (stationary)

1. Binary white noise

$$X(t) = \begin{cases} +1 & \pi \\ -1 & 1-\pi \end{cases}$$

$c_{XX}(u) = \pi(1-\pi)$ $u=0$ and $=0$ $u \neq 0$

$$\boxed{c_{YX}(v) = \sum_u a(u)\, c_{XX}(v-u)}$$

$$c_{YX}(v) = \sigma_X^2\, a(v)$$

2. Gaussian white noise.

Good for nonlinear systems

3. Mixture of white noises

still white

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4. Band-limited white noise

$A(\lambda) = 0 \quad 0 < \lambda < \lambda \quad \text{and} \quad \lambda < \lambda$

Spread spectrum

$\{W(t)\}$ white

$X(t) = W(t) \cos \lambda_0 t$

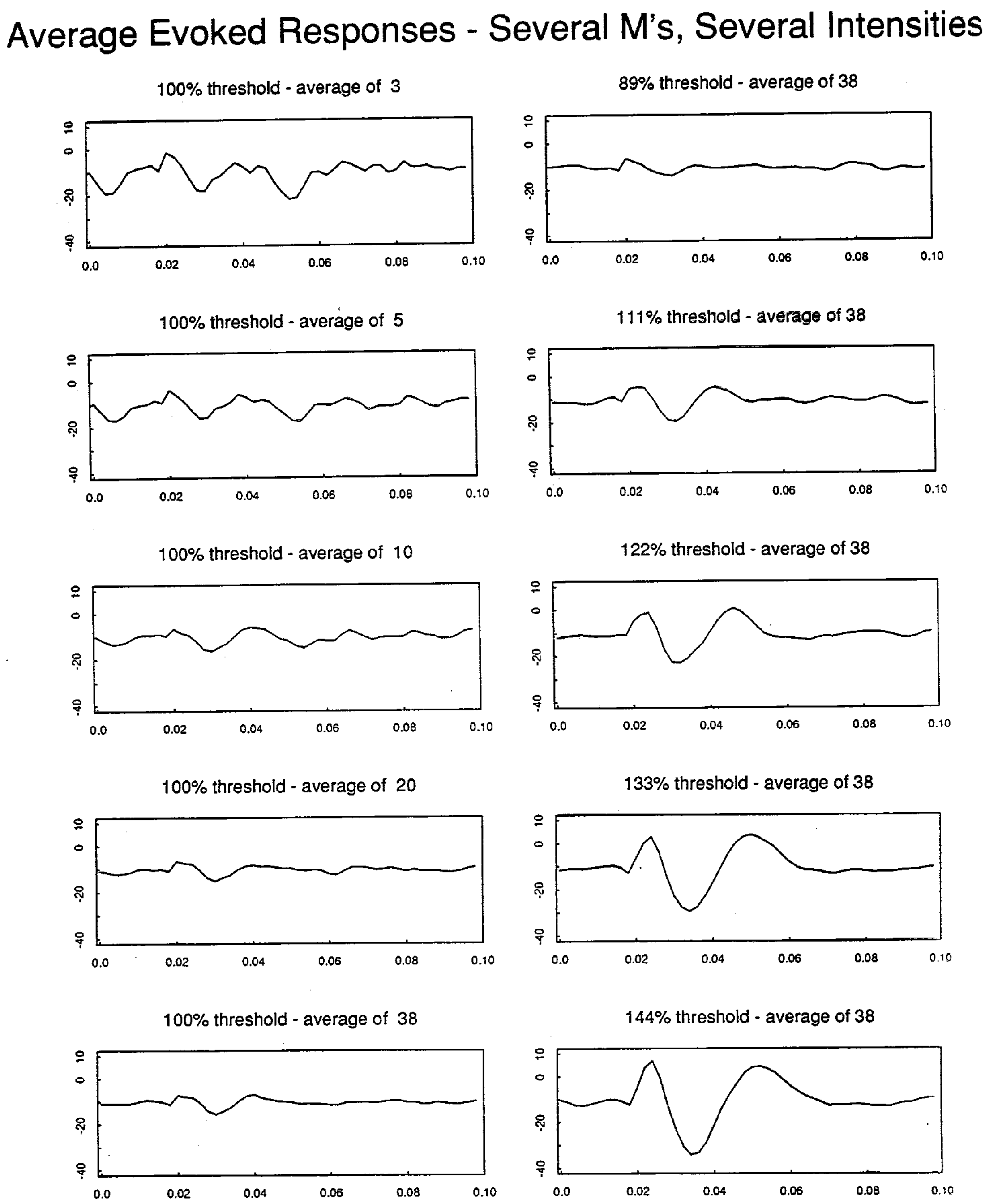

FIG. 22. *The various graphs here are meant to show the effects of changing the number of responses averaged (left column) and the strength of stimulus applied (right column) for data such as that of Figure* 21.