(1)

April 13, 2009

Interpretation of spectrum estimate
aliasing, Nyquist frequency

Stationary $\{X(t), -\infty < t < \infty\}$

$$c_{XX}(u) = \text{cov}\{X(t+u), X(t)\} \quad -\infty < t, t+u < \infty$$

$$g_{XX}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c_{XX}(u) \exp\{-i\lambda u\} du$$

$$-\infty < \lambda < \infty$$

$$g_{XX}(-\lambda) = g_{XX}(\lambda)$$

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda} dZ_X(\lambda) \; ; \; \text{cov}\{dZ_X(\lambda), dZ_X(\mu)\} = \delta(\lambda-\mu) f_{XX}(\lambda) d\lambda d\mu$$

"sampled" series $X(jh)$ $j = 0, \pm 1, \pm 2, \ldots$

sampling interval h

April 13, 2009

$$X_j = X(jh) = \int_{-\infty}^{\infty} e^{ijh\lambda} dZ_X(\lambda)$$

Cramér

$$e^{ijh(\lambda + \frac{2\pi}{h})} = e^{ijh\lambda + ij2\pi} = e^{ijh\lambda}$$

$$\int_{-\infty}^{\infty} e^{ijh\lambda} dZ_X(\lambda) = \int_{-\pi/h}^{\pi/h} e^{ijh\lambda} d\left[\sum_{k=-\infty}^{\infty} dZ_X\left(\lambda + \frac{2\pi k}{h}\right)\right]$$

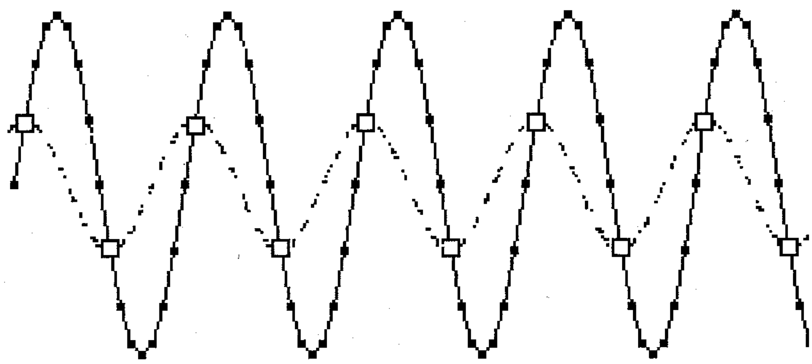
$$d\tilde{Z}_X(\lambda) = \sum_{k=-\infty}^{\infty} dZ_X\left(\lambda + \frac{2\pi k}{h}\right)$$

$$f_{XX}(\lambda) = \sum_{k=-\infty}^{\infty} g_{XX}\left(\lambda + \frac{2\pi k}{h}\right)$$

$$-\pi/h < \lambda \leq \pi/h$$

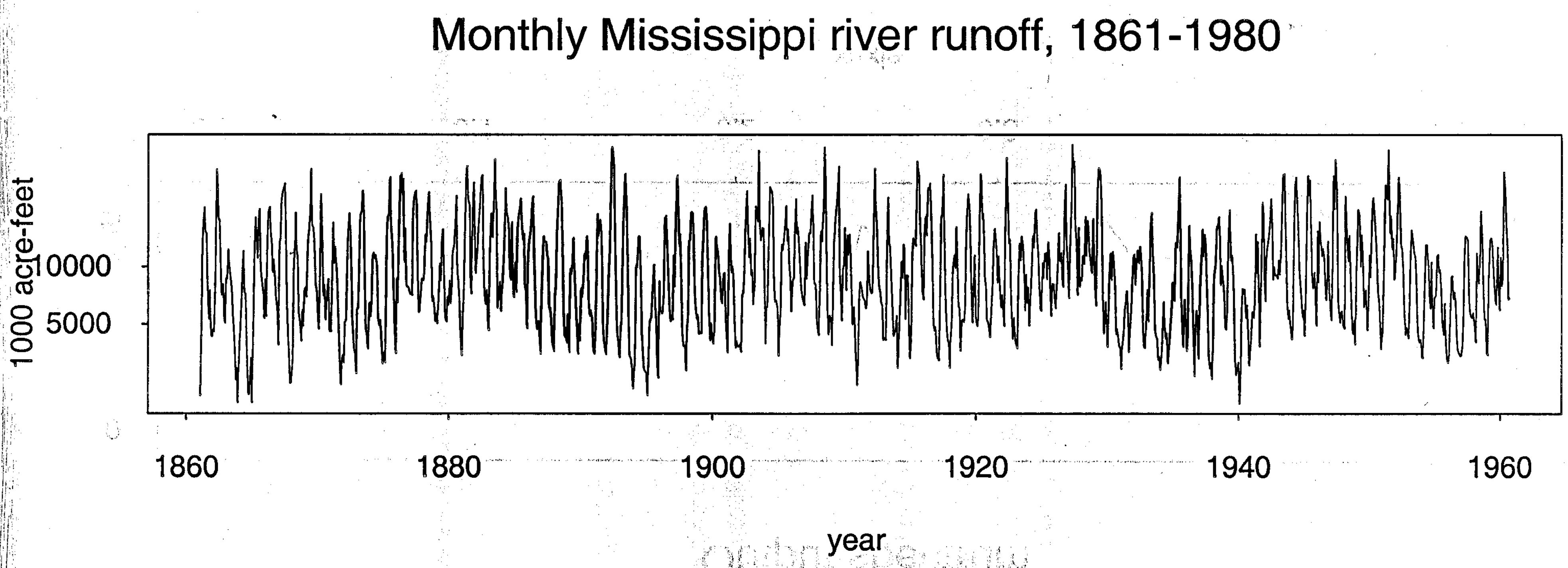
<u>aliases</u> $\lambda + \frac{2\pi k}{h}, \; -\lambda + \frac{2\pi k}{h} \quad k = 0, \pm 1, \ldots$

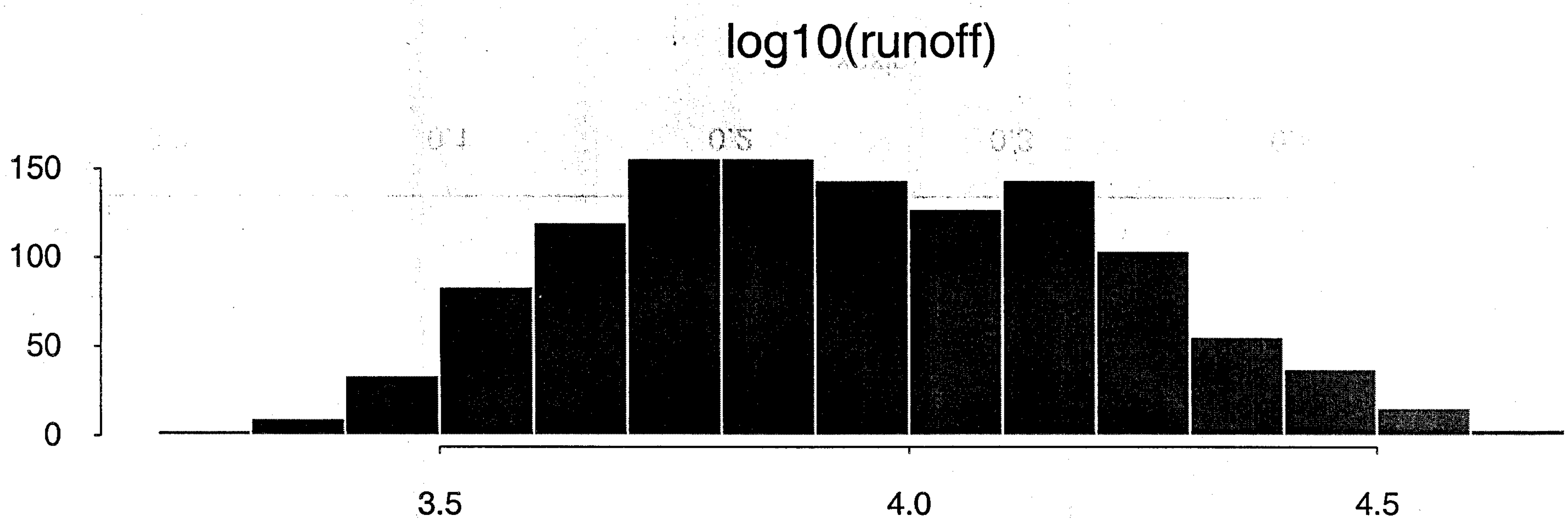
<u>Nyquist frequency</u> π/h



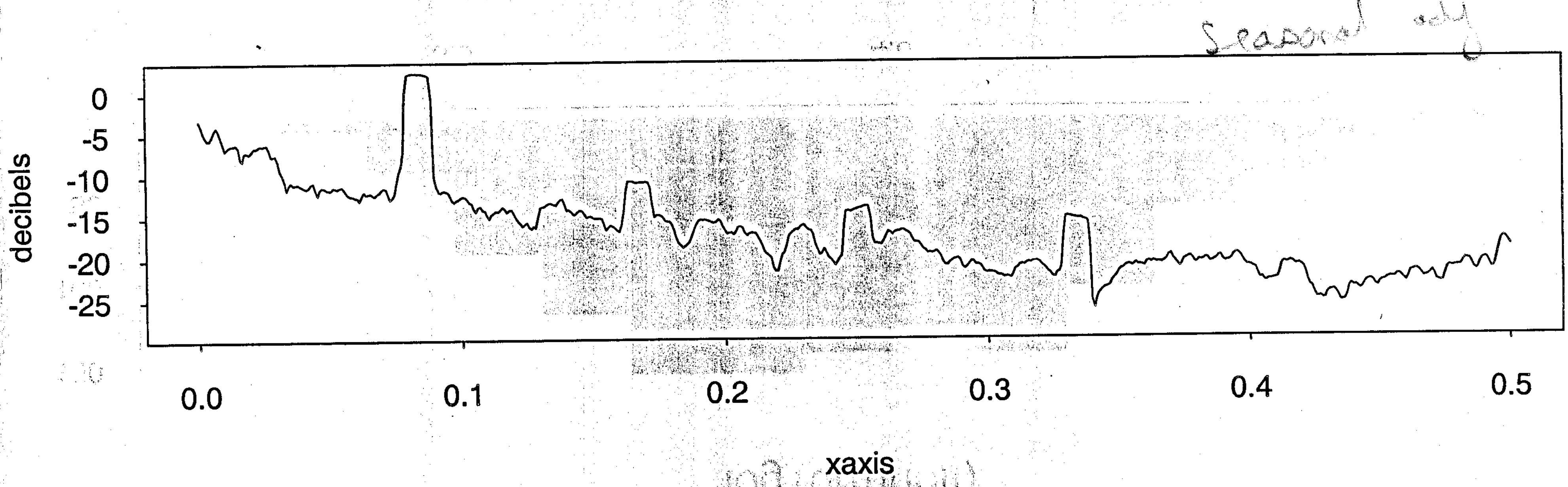
```
postscript(file="Po.ps",hor=T)
par(mfrow=c(2,1))
Junk<-scan("../../Data/mississ")
Junkl<-length(Junk)
xaxis<-1861+(1:Junkl)/12
plot(xaxis,Junk,type="l",main="Monthly Mississippi river runoff, 1861-1960
",xlab="year",ylab="1000 acre-feet",las=1,log="y")
hist(log10(Junk),main="log10(runoff)",xlab="",ylab="",las=1)
Junk1<-ts(data=log10(Junk),start=1861+1/12,frequency=12)
Junk2<-sabl(Junk1,power=1,seasonal=15,trend=201)
y<-Junk2$irregular;u<-log10(Junk)
x<-cbind(u,y)
junk<-spec.pgram(x,spans=13,taper=0,detrend=F,demean=T)
xaxis<-junk$freq
#par(mfrow=c(2,2))
ylim<-range(junk$spec)
plot(xaxis,junk$spec[,1],main="Input spectrum",las=1,type="l",ylab="decibe
ls",ylim=ylim)
plot(xaxis,junk$spec[,2],main="Output spectrum",las=1,type="l",ylab="decib
els",ylim=ylim)
junk1<-sqrt(junk$coh*10^((junk$spec[,2]-junk$spec[,1])/10))
plot(xaxis,junk1,type="l",las=1,main="Gain",ylab="gain",xlab="frequency",l
og="y")
plot(xaxis,junk$phase,type="l",las=1,main="Phase",xlab="frequency",ylab="p
hase")
plot(xaxis,junk$coh,type="l",ylim=c(0,1),las=1,main="Coherence",xlab="freq
uency",ylab="coherence")
graphics.off()
q()
```

```
stats::decompose        Classical Seasonal Decomposition by Moving
                        Averages
stats::monthplot        Plot a Seasonal or other Subseries from a Time
                        Series
stats::stl              Seasonal Decomposition of Time Series by Loess
```

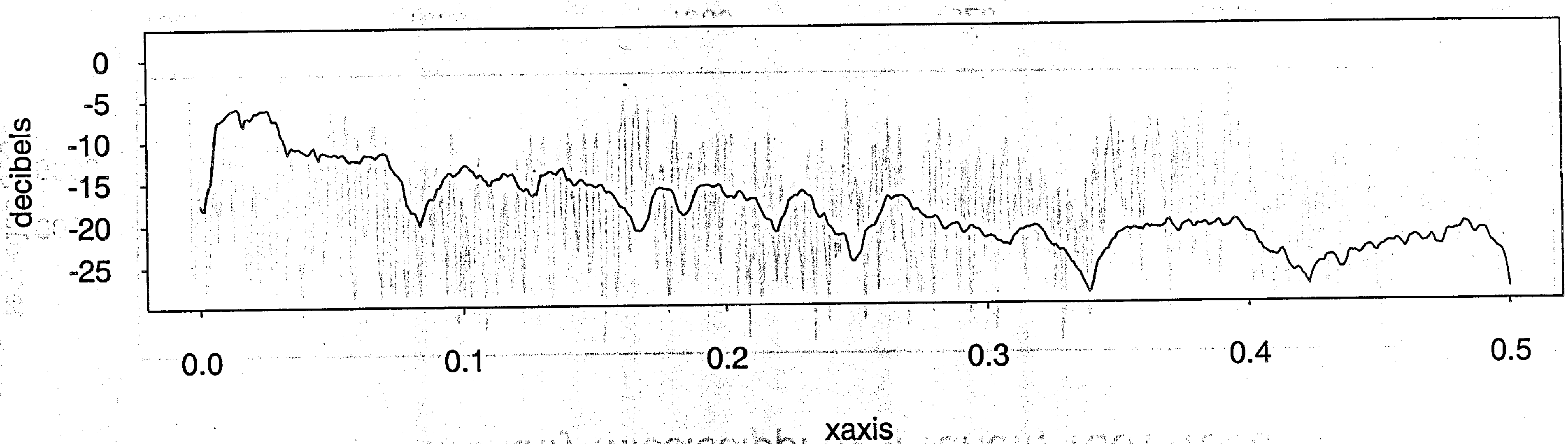

Monthly Mississippi river runoff, 1861-1980
1000 acre-feet
5000
10000
1860
1880
1900
1920
1940
1960
year

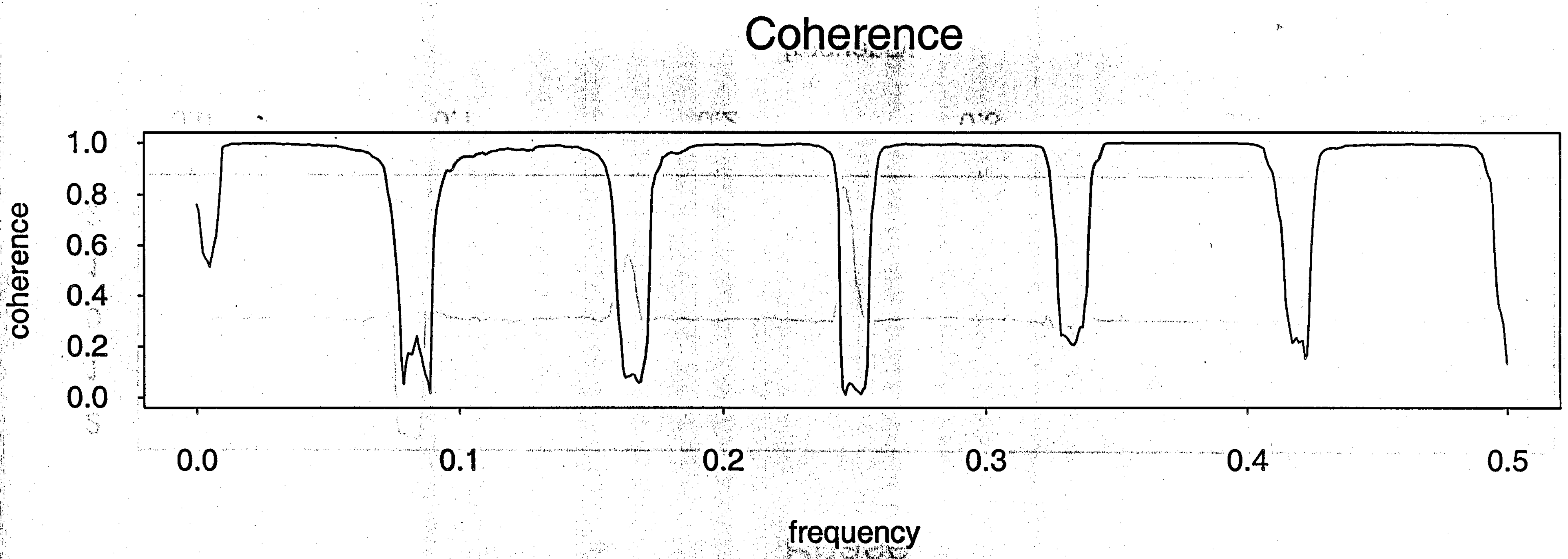

log10(runoff)
0
50
100
150
3.5
4.0
4.5

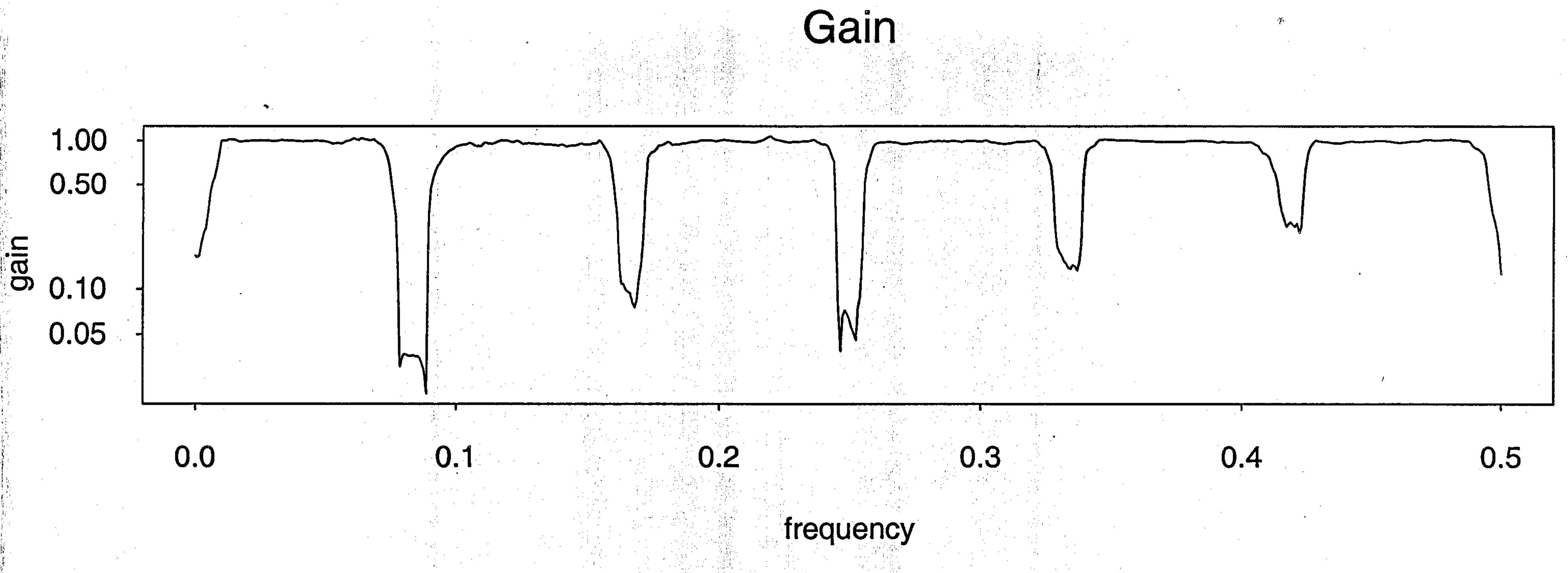
Input spectrum

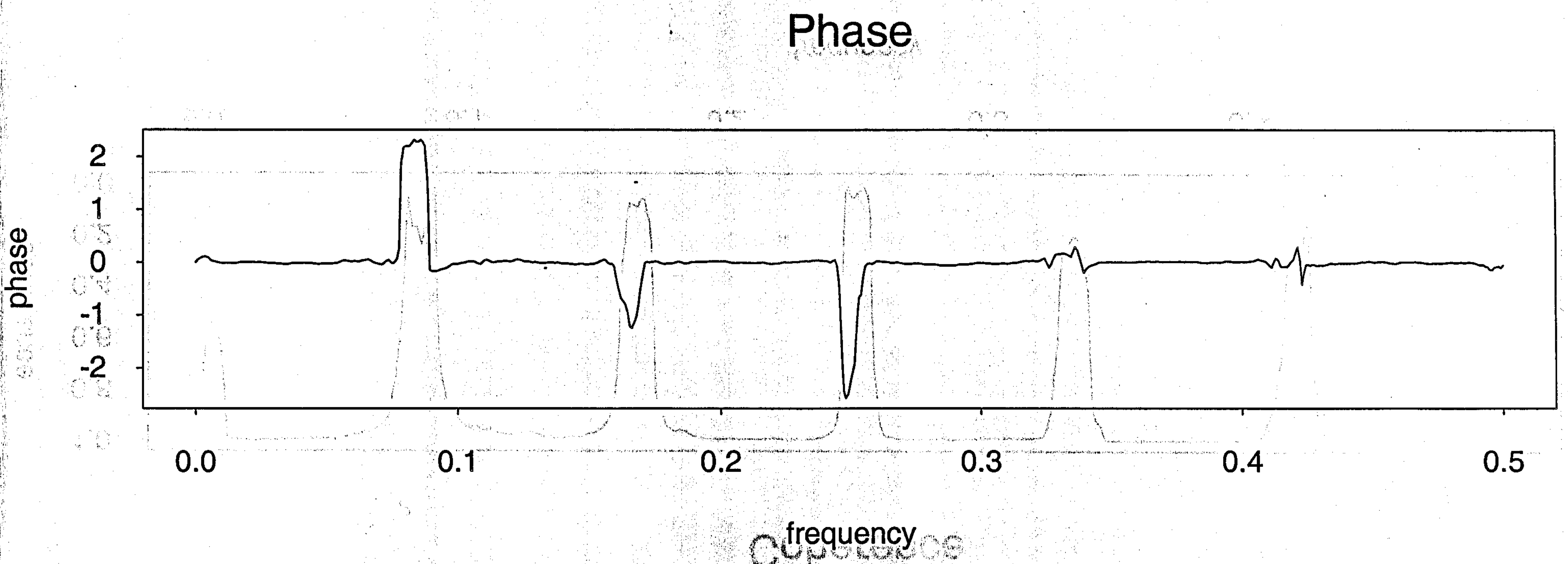


Output spectrum




Coherence
coherence
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
frequency


Gain
gain
1.00
0.50
0.10
0.05
0.0
0.1
0.2
0.3
0.4
0.5
frequency


Phase
phase
2
1
0
-1
-2
0.0
0.1
0.2
0.3
0.4
0.5
frequency

24 April 03

Advantages of white noise.

1. Simplicity of interpretation
2. Simplicity of computation
3. Efficiency

12 April 2009

White noise - another motivation

Consider case of $p=2$

$$\underset{n\times 1}{\underset{\sim}{Y}} = \beta_1 \underset{n\times 1}{\underset{\sim}{x_1}} + \beta_2 \underset{n\times 1}{\underset{\sim}{x_2}} + \underset{n\times 1}{\underset{\sim}{\varepsilon}}$$

$$\underset{n\times 1}{Y} = \underset{n\times 2}{X} \underset{n\times 1}{\beta} + \varepsilon$$

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

$$\operatorname{var} \hat{\beta} = (X^T X)^{-1} \sigma^2$$

Suppose $x_j^T x_j = C_j$, then

$$E|\hat{\beta}_j - \beta_j|^2 \geq 1/C_j$$

with minimum when $x_j^T x_k = 0 \quad j \neq 2$

12 April 2009

White noise – another motivation

Consider case of $p=2$

$$\underset{n\times 1}{\underset{\sim}{Y}} = \beta_1 \underset{n\times 1}{\underset{\sim}{x}_1} + \beta_2 \underset{n\times 1}{\underset{\sim}{x}_2} + \underset{n\times 1}{\underset{\sim}{\varepsilon}}$$

$$\underset{n\times 1}{Y} = \underset{n\times 2}{X} \underset{n\times 1}{\beta} + \varepsilon$$

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

$$\operatorname{var} \hat{\beta} = (X^T X)^{-1} \sigma^2$$

Suppose $x_j^T x_j = C_j$, then

$$E|\hat{\beta}_j - \beta_j|^2 \geq 1/C_j$$

with minimum when $x_j^T x_k = 0 \quad j \neq 2$

12 April 2009

$$X^T X = \begin{bmatrix} x_1^T x_1 & x_1^T x_2 \\ x_2^T x_1 & x_2^T x_2 \end{bmatrix} = \begin{bmatrix} C_1 & \theta \\ \theta & C_2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d/D & -b/D \\ -c/D & a/D \end{bmatrix} \quad D = ad - bc$$

$$(X^T X)^{-1} = \begin{bmatrix} C_2/(C_1 C_2 - \theta^2) & -\theta/(C_1 C_2 - \theta^2) \\ -\theta/(C_1 C_2 - \theta^2) & C_1/(C_1 C_2 - \theta^2) \end{bmatrix}$$

$\mathrm{Var}\,\hat{\beta}_j$ min at $\theta = 0$, $x_1^T x_2 = 0$

value $1/C_j$

C. R. Rao (1973) Linear Statistical Inference and Its Applications Second edition

p. 236

12 April 2009

Estimating impulse response by least squares

$$\begin{bmatrix} \vdots \\ Y_{t-1} \\ Y_t \\ Y_{t+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & & \vdots \\ X_{t-1} & X_{t-2} & & X_{t-p-1} \\ X_t & X_{t-1} & & X_{t-p} \\ X_{t+1} & X_t & & X_{t-p+1} \\ \vdots & & & \vdots \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix}$$

Want $[\ldots X_{t-1}\ X_t\ X_{t+1}\ \ldots]$

orthogonal to

$$[\ldots X_{t-u-1}\ X_{t-u}\ X_{t-u+1}\ \ldots]$$

for $u \neq 0$

16 April 03

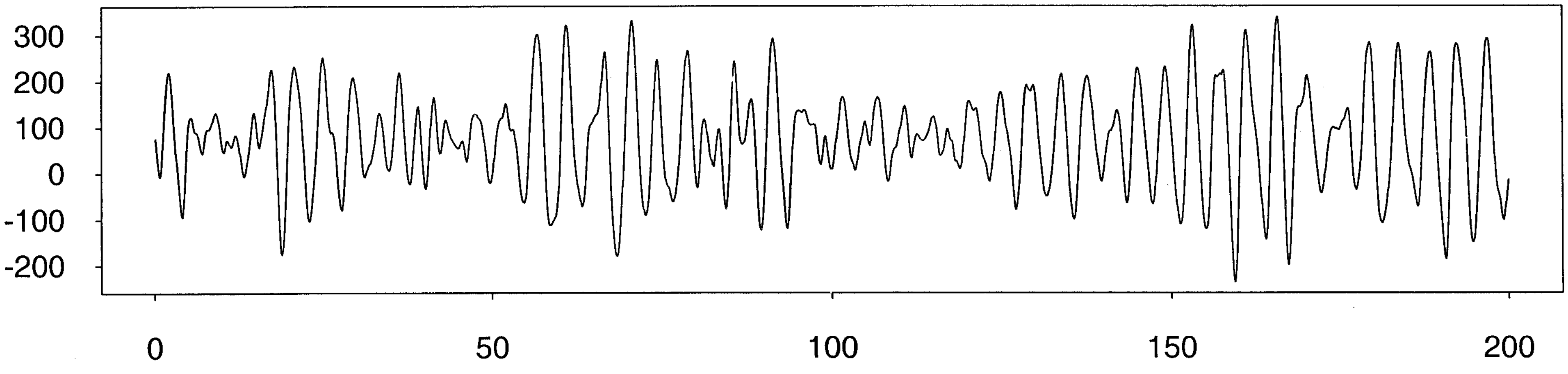
Importance of prefiltering

$$Y(t) = \alpha X(t-v) + \epsilon(t)$$

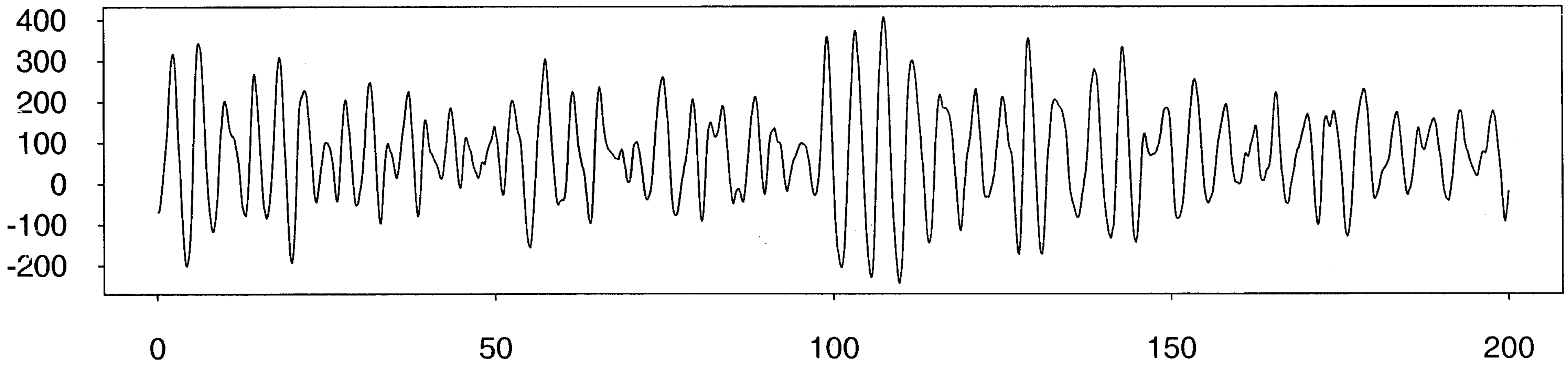
$$E f^T_{YX}(\lambda) \simeq \frac{1}{2\pi T}\frac{1}{m}\sum_j A\left(\frac{2\pi j}{T}\right)\left|d^T_X\left(\frac{2\pi j}{T}\right)\right|^2$$

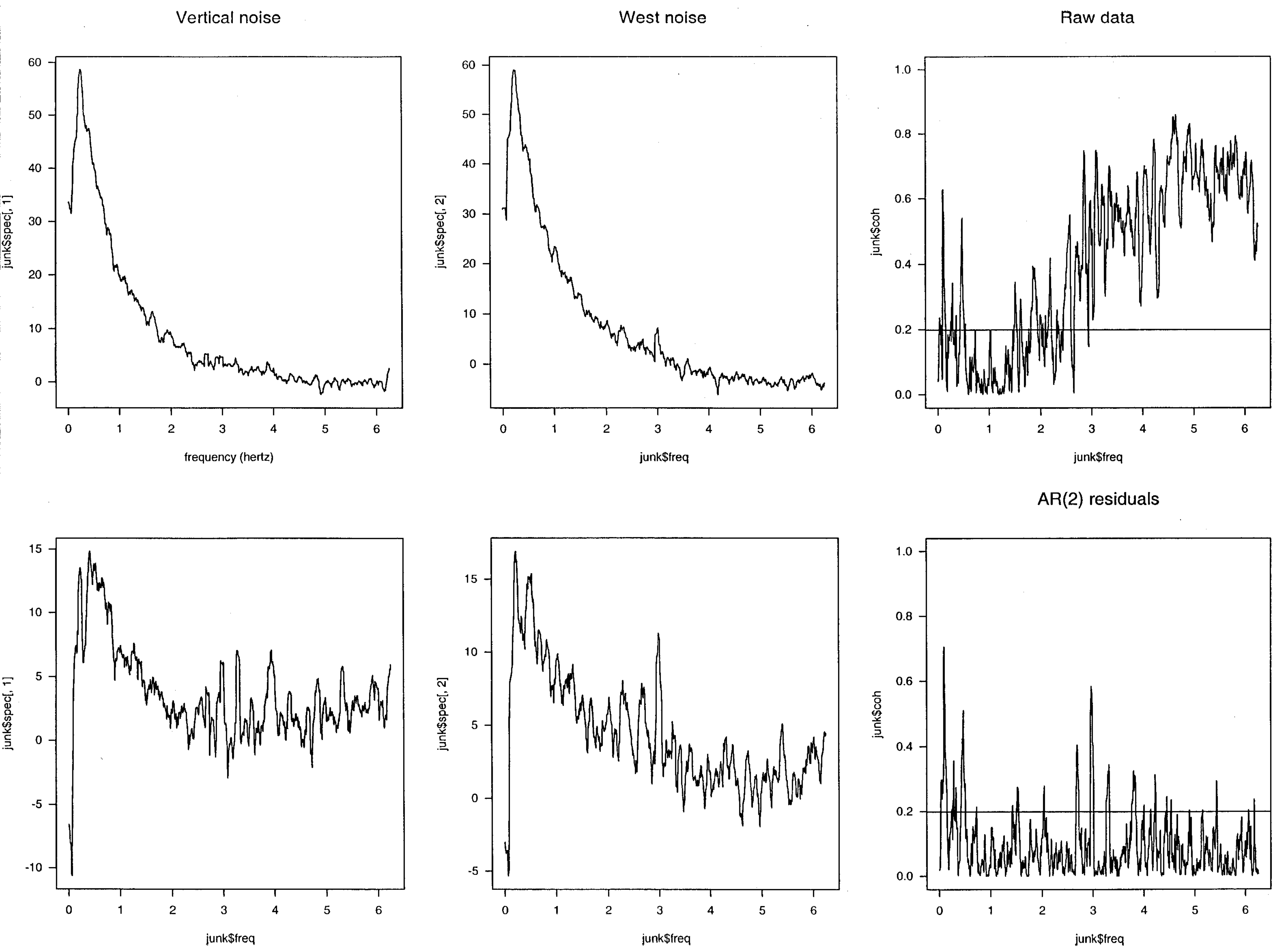
$$\simeq \frac{1}{2\pi T}\frac{1}{m}\sum_j e^{i\frac{2\pi j v}{T}}\left|d^T_X\left(\frac{2\pi j}{T}\right)\right|^2$$

Vertical component seismic noise at Berkeley



West component




Vertical noise
West noise
Raw data
AR(2) residuals
junk$spec[, 1]
junk$spec[, 2]
junk$coh
frequency (hertz)
junk$freq

24 April 03

Bivariate stationary $\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}$ $t = 0, \pm 1, \ldots$

Determine μ, $a(\cdot)$, $A(\cdot)$ to minimize

$$E\left|Y(t) - \mu - \sum_u a(t-u) X(u)\right|^2 \qquad (*)$$

$\frac{\partial}{\partial \mu}$: $\quad E\,Y(t) = \mu + A(0)\,E\,X(t)$

$$(*) = \int [f_{YY}(\lambda) - A(\lambda) f_{XY}(\lambda) - f_{YX}(\lambda)\overline{A(\lambda)} + A(\lambda) f_{XX}(\lambda) \overline{A(\lambda)}]\, d\lambda$$

$$A(\lambda) = f_{YX}(\lambda) f_{XX}(\lambda)^{-1} \qquad \text{(add \& subtract)}$$

Minimum

$$\int [1 - |R(\lambda)|^2] f_{YY}(\lambda)\, d\lambda$$

24 April 03

Might define

$$\epsilon(t) = Y(t) - \mu - \sum_u a(u)\, X(t-u)$$

$$E\,\epsilon(t) = 0$$

$$f_{\epsilon\epsilon}(\lambda) = [1 - |R(\lambda)|^2]\, f_{YY}(\lambda)$$

$$\{\epsilon(t)\} \perp \{X(t)\}$$

If $\left\{\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}\right\}$ Gaussian, then

$$E\{Y(t) \mid X\} = \mu + \sum_u a(t-u)\, X(u)$$

$$\operatorname{Var}\{Y(t) \mid X\} = \int [1 - |R(\lambda)|^2]\, f_{YY}(\lambda)\, d\lambda$$

with above μ, $A(\cdot)$

21 April 03

Instrumental series.

$$Y(t) = \mu + \sum_u a(t-u)\mathcal{X}(u) + \epsilon(t)$$

but $X(t) = \mathcal{X}(t) + \eta(t)$

Suppose can find $\{Z(t)\}$ correlated with $\{\mathcal{X}(t)\}$ but not $\{\epsilon(t)\}, \{\eta(t)\}$

Have $f_{YZ}(\lambda) = A(\lambda) f_{XZ}(\lambda)$

magnetotellurics

21 April 03

Matched filter

$\{Y(t)\}$: signal

$\{\eta(t)\}$: noise

Observe $X(t) = Y(t) + \eta(t)$

$$f_{YX}(\lambda) = f_{YY}(\lambda)$$

$$f_{XX}(\lambda) = f_{YY}(\lambda) + f_{\eta\eta}(\lambda)$$

$$A(\lambda) = f_{YY}(\lambda)\left[f_{YY}(\lambda) + f_{\eta\eta}(\lambda)\right]^{-1}$$

$\simeq 1$ $\quad f_{\eta\eta}(\lambda) = 0$

$\simeq 0$ $\quad f_{\eta\eta}(\lambda)$ large

$\dfrac{f_{YY}(\lambda)}{f_{\eta\eta}(\lambda)}$: signal to noise ratio at frequency λ

N. Wiener