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## Regression in the complex case.

$Y_j = \beta x_j + \epsilon_j$ , $j = 1, \ldots, n$

$x$'s fixed

$\epsilon_j$ : $IN^C(0, \sigma^2)$

$EY_j = \beta x_j$ $\quad$ $\mathrm{var}\, Y_j = \sigma_j^2$

LSE $\hat{\beta} = \sum_j Y_j \bar{x}_j / \sum_j |x_j|^2$

$= \beta + \sum_j \epsilon_j \bar{x}_j / \sum_j |x_j|^2$

$\hat{\beta} \sim N^C(\beta, \sigma^2/\sum_j |x_j|^2)$

$\hat{\sigma}^2 = \sum_j |Y_j - \hat{\beta} x_j|^2/(n-1)$

$\sim \sigma^2 \chi^2_{2(n-1)}/2(n-1)$

$\hat{\beta} \perp\!\!\!\perp \hat{\sigma}^2$

$1 - |\hat{R}|^2 = \sum_j |Y_j - \hat{\beta} x_j|^2 / \sum_j |Y_j|^2$

$0 \le |\hat{R}|^2 \le 1$

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## Cross-spectral analysis

Separate regression at each frequency

Model

$$Y(t) = \mu + \sum_u a(u) X(t-u) + \epsilon(t) \qquad (*)$$

$\{\epsilon(\cdot)\}$ 0-mean, stationary

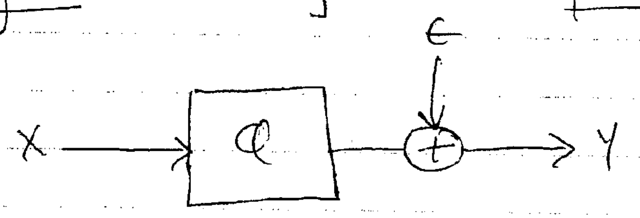
$$EY(t) = \mu + \sum_u a(u) X(t-u)$$

nonstationary

$\{a(\cdot)\}$ impulse response

$$A(\lambda) = \sum_u e^{-i\lambda u} a(u) \qquad \underline{\text{transfer function}}$$

$|A(\lambda)|$: <u>gain</u> $\quad$ $\arg A(\lambda)$: <u>phase</u>




## System identification

Estimate $\{a(\cdot)\}$, $\{A(\cdot)\}$, $f_{\epsilon\epsilon}(\lambda)$ from data $(X(t), Y(t))$, $t = 0, \ldots, T-1$

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Pick $n$ distinct $\frac{2\pi j}{T}$ near $\lambda$

FT ⊛

$$d_Y^T(\lambda) \simeq A(\lambda) d_X^T(\lambda) + d_\epsilon^T(\lambda)$$

$$d_Y^T\left(\frac{2\pi j}{T}\right) \simeq A(\lambda) d_X^T\left(\frac{2\pi j}{T}\right) + d_\epsilon^T\left(\frac{2\pi j}{T}\right)$$

cp. $Y_j = \beta x_j + \epsilon_j$

CLT for $d_\epsilon^T\left(\frac{2\pi j}{T}\right)$ : $IN^C(0, 2\pi T f_{\epsilon\epsilon}(\lambda))$

$Y_j \sim d_Y^T\left(\frac{2\pi j}{T}\right)$ ; $x_j \sim d_X^T\left(\frac{2\pi j}{T}\right)$ ; $\epsilon_j \sim d_\epsilon^T\left(\frac{2\pi j}{T}\right)$

$\beta \sim A(\lambda)$ ; $\sigma^2 \sim 2\pi T f_{\epsilon\epsilon}(\lambda)$

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$$A^T(\lambda) = \sum_j d_Y^T\left(\frac{2\pi j}{T}\right)\overline{d_X^T\left(\frac{2\pi j}{T}\right)} \Big/ \sum_j \left|d_X^T\left(\frac{2\pi j}{T}\right)\right|^2$$

$$\sim N^C\left(A(\lambda),\ 2\pi T f_{\epsilon\epsilon}(\lambda) \Big/ \sum_j \left|d_X^T\left(\frac{2\pi j}{T}\right)\right|^2\right)$$

$$||$$

$$f_{\epsilon\epsilon}^T(\lambda) = \sum_j \left|d_Y^T\left(\frac{2\pi j}{T}\right) - A^T(\lambda)\, d_X^T\left(\frac{2\pi j}{T}\right)\right|^2 \Big/ 2\pi T(m-1)$$

$$\sim f_{\epsilon\epsilon}(\lambda)\, \chi^2_{2(m-1)} / 2(m-1)$$

$$\sum_j \left|d_X^T\left(\frac{2\pi j}{T}\right)\right|^2 \sim 2\pi T m\, f_{XX}^T(\lambda)$$

$$|R_{YX}^T(\lambda)|^2 = 1 - f_{\epsilon\epsilon}^T(\lambda) / f_{YY}^T(\lambda)$$

$$= |f_{YX}^T(\lambda)|^2 / f_{XX}^T(\lambda)\, f_{YY}^T(\lambda)$$

$$E|R_{YX}^T(\lambda)|^2 \sim 1/(m-1)$$

$100\alpha\%$ pt $\quad 1 - (1-\alpha)^{1/(m-1)}$

$$f_{\epsilon\epsilon}^T(\lambda) = [1 - |R_{YX}^T(\lambda)|^2]\, f_{YY}^T(\lambda)$$

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$$f^T_{YX}(\lambda) = \frac{1}{2\pi T}\frac{1}{m}\sum_j d^T_Y\left(\frac{2\pi j}{T}\right)\overline{d^T_X\left(\frac{2\pi j}{T}\right)}$$

Plot

$$|R^T_{YX}(\lambda)|^2$$

$$\log|A^T(\lambda)|$$

$$\arg A^T(\lambda)$$

$$AV(\log|A^T(\lambda)|) \sim \frac{1}{2m}\left[|R^T_{YX}(\lambda)|^{-2} - 1\right]$$

See DRB book Chapter 6

$m$ can be $m_T \to \infty$, $\frac{m_T}{T} \to 0$

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## Splitting approach

Suppose $T = LV$ having in mind

$V \to \infty$, $L$ fixed

Let $j = 1, \dots, L$

Compute

$$y_j = d_Y^V(\lambda, j) = \sum_{v=0}^{V-1} e^{-i\lambda v} Y(v + \overline{j-1}\,V)$$

$$x_j = d_X^V(\lambda, j) =$$

Better for higher-order spectra

$L$ can be $L_T$

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# Importance of prefiltering

$$Y(t) = \alpha X(t-v) + \epsilon(t)$$

$$E f_{YX}^{T}(\lambda) \simeq \frac{1}{2\pi T} \frac{1}{m} \sum_j A\left(\frac{2\pi j}{T}\right) \left|d_X^T\left(\frac{2\pi j}{T}\right)\right|^2$$

$$\simeq \frac{1}{2\pi T} \frac{1}{m} \sum_j e^{i\frac{2\pi j v}{T}} \left|d_X^T\left(\frac{2\pi j}{T}\right)\right|^2$$