(a)

State setup.

Deals with many difficulties
- missing values
- fitting ARMAs by mle
- irregularly spaced data
- aggregation / disaggregation
- measurement error
- outliers
- structural breaks
- prediction, smoothing
- nonlinearity
- discrete / continuous time
- explanatories
- seasonals
- time changing

consulting

Proceeds via recursions

R.H. Shumway and D.S. Stoffer (2006), _Time Series Analysis and its Applications, with R Examples_, second edition. Springer

The State space approach.

State.

summary of the past behavior of the system

taken together with the future system inputs determines all future states and system outputs

output is a function of the current state and current input values only

x_t, the state is a Markov process (everything is)

Examples.

Model 1. Nonlinear

$$x_t = f(x_{t-1}) + w_t \quad , \text{ state}$$

$$y_t = h(x_t) + v_t \quad , \text{ measurement}$$

$\{v_t\}$, $\{w_t\}$ white noise

$f(x_0)$ given , initial

Perhaps input $\{u_t\}$

(ii)

Model 2. Linear

$$x_{t+1} = A_t x_t + K_t w_t$$

$$y_t = C_t x_t + v_t$$

$$f(x_0)$$

vectors, matrices

Model 3. General

$$x_{t+1} \sim q(x_{t+1} \mid x_t)$$

$$y_t \sim r(y_t \mid x_t)$$

$$f(x_0)$$

Model 4. Discrete-valued t.s.

$$x_{t+1} = A_t x_t + K_t w_t$$

$$y_t \sim \mathrm{Dist}(x_t)$$

May motivate by both linearity and gaussianity.

Model identification.

$$Z^t = \{y_1, \ldots, y_t\} \quad , \text{history}$$

$$L(\theta) = f(y_1, \ldots, y_n \mid \theta) \quad , \text{likelihood}$$

$$= \prod_{t=1}^{n} f(y_t \mid y_1, \ldots, y_{t-1})$$

$$= \prod_{t=1}^{n} f(y_t \mid Z^{t-1})$$

Linear gaussian model.

$$f(y_t \mid Z^{t-1}) = \frac{1}{\sqrt{2\pi r_t}} \exp\left\{ - \frac{(y_t - c_t x_{t|t-1})^2}{2 r_t} \right\}$$

r_t etc. via "Kalman filter"

Can employ mle

General filtering and smoothing.
non Gaussian

One step ahead prediction.

$$f(x_t \mid Z^{t-1}) = \int f(x_t, x_{t-1} \mid Z^{t-1})\, dx_{t-1}$$

$$= \int f(x_t \mid x_{t-1}, Z^{t-1})\, p(x_{t-1} \mid Z^{t-1})\, dx_{t-1}$$

$$= \int q(x_t \mid x_{t-1})\, f(x_{t-1} \mid Z^{t-1})\, dx_{t-1}$$

Filtering.

$$f(x_t \mid Z^t) = f(x_t \mid y_t, Z^{t-1})$$

$$= \frac{f(y_t \mid x_t, Z^{t-1})\, f(x_t \mid Z^{t-1})}{f(y_t \mid Z^{t-1})}$$

$$= \frac{r(y_t \mid x_t)\, f(x_t \mid Z^{t-1})}{f(y_t \mid Z^{t-1})}$$

$$f(y_t \mid Z^{t-1}) = \int r(y_t \mid x_t)\, f(x_t \mid Z^{t-1})\, dx_t$$

Smoothing.

$$f(x_t \mid Z^n) = \int f(x_t, x_{t+1} \mid Z^n)\, dx_{t+1}$$

$$= \int f(x_{t+1} \mid Z^n)\, f(x_t \mid x_{t+1}, Z^n)\, dx_{t+1}$$

$$= \int f(x_{t+1} \mid Z^n)\, f(x_t \mid x_{t+1}, Z^t)\, dx_{t+1}$$

$$= f(x_t \mid Z^t) \int \frac{f(x_{t+1} \mid Z^n)\, f(x_{t+1} \mid x_t, Z^t)}{f(x_{t+1} \mid Z^t)}\, dx_{t+1}$$

$$= f(x_t \mid Z^t) \int \frac{f(x_{t+1} \mid Z^t)\, g(x_{t+1} \mid x_t)}{f(x_{t+1} \mid Z^t)}\, dx_{t+1}$$

Nonlinear State-Space Models.

§5.3 in text

$$x(t+1) = f(t, x(t), u(t), w(t); \theta)$$

$$y(t) = h(t, x(t), u(t), v(t); \theta)$$

finite dimensional

$u(t)$, $v(t)$: sequences of independents

The predictor

$$\hat{y}(t|\theta) = g(t, Z^{t-1}; \theta)$$

is now a nonlinear function of past observations

prediction error

$$\varepsilon(t, \theta) = y(t) - g(t, Z^{t-1}; \theta)$$

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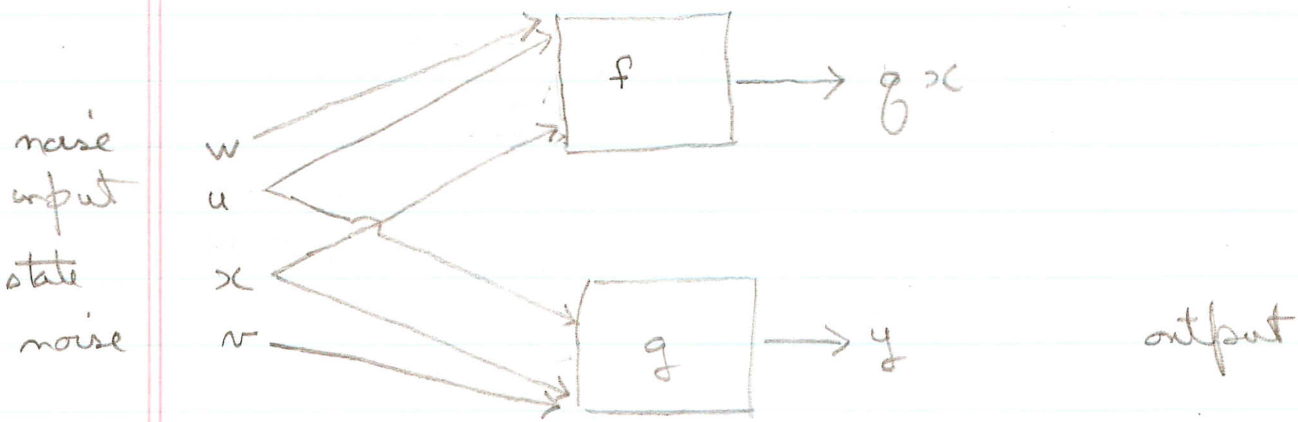

Examples.

1. Random walk + noise

$$x_t = x_{t-1} + w_t$$

$$y_t = x_t + v_t$$

2. Case of drift. (local linear trend)

$$\beta_t = \beta_{t-1} + w_{2,t}$$

$$\mu_t = \mu_{t-1} + \beta_{t-1} + w_{1,t}$$

$$y_t = x_t + v_t$$

$$x_t = [\mu_t, \beta_t]'$$

3. Local seasonal.

$$\beta_t = \beta_{t-1} + w_{2,t}$$

$$\mu_t = \mu_{t-1} + \beta_{t-1} + w_{1,t}$$

$$i_t = \sum_{j=1}^{s-1} i_{t-j} + w_{3,t}$$

$$x_t = [\mu_t \ \beta_t \ i_t \ \dots \ i_{t-s+1}]'$$

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4. AR(2).

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + v_t$$

$$Y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t$$

$$x_t = \begin{bmatrix} \varphi_1 & 1 \\ \varphi_2 & 0 \end{bmatrix} x_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_t$$

(1)

ARMA(p, q) as state space

$$\sum_{u=0}^{p} a(u)\, Y(t-u) = \sum_{v=0}^{q} b(v)\, \varepsilon(t-v)$$

$$Y(t) = [1 \quad \underset{\sim}{0}'_{r-1}]\ \underset{\sim}{S}(t)$$

$$\underset{\sim}{S}(t) = \underset{\sim}{F}\ \underset{\sim}{S}(t-1) + \underset{\sim}{\Psi}\, \varepsilon(t)$$

$$\underset{\sim}{F} = \left[\begin{array}{c|c} -a(1) & \\ \vdots & \underset{\sim}{I}_{r-1} \\ -a(r-1) & \\ \hline -a(r) & \underset{\sim}{0}'_{r-1} \end{array}\right] \qquad \underset{\sim}{\Psi} = \begin{bmatrix} 1 \\ b(1) \\ \vdots \\ b(r-1) \end{bmatrix} \qquad \begin{array}{l} b(0) = 1 \\ r = \max(p, q+1) \end{array}$$

To define $\underset{\sim}{S}(t)$

$$Y(t) = S_1(t)$$

$$S_r(t) = -a(r)\, Y(t-1) + b(r-1)\, \varepsilon(t)$$

$$\begin{aligned} S_{r-1}(t) &= -a(r-1)\, Y(t-1) + S_r(t-1) + b(r-2)\, \varepsilon(t) \\ &= -a(r-1)\, Y(t-1) - a(r)\, Y(t-2) + b(r-2)\, \varepsilon(t) + b(r-1)\, \varepsilon(t-1) \end{aligned}$$

$$\begin{aligned} S_{r-2}(t) &= -a(r-2)\, Y(t-1) + S_{r-1}(t-1) + b(r-3)\, \varepsilon(t) \\ &= -a(r-2)\, Y(t-1) - a(r-1)\, Y(t-2) - a(r)\, Y(t-3) + b(r-3)\, \varepsilon(t) \\ &\quad + b(r-2)\, \varepsilon(t-1) + b(r-1)\, \varepsilon(t-2) \end{aligned}$$

$$\begin{aligned} S_{r-j}(t) &= -a(r-j)\, Y(t-1) - a(r-j+1)\, Y(t-2) - a(r-j+2)\, Y(t-3) \\ &\quad - \dots + b(r-j-1)\, \varepsilon(t) + \dots \end{aligned}$$

$j = r-1$

$$S_1(t) = -a(1)\, Y(t-1) - a(2)\, Y(t-2) - a(3)\, Y(t-3) - \dots + b(0)\, \varepsilon(t) + \dots$$