# Introduction to Time Series Analysis. Lecture 17.

1. Spectral density: Facts and examples.

2. Spectral distribution function.

3. Wold's decomposition.

## A periodic time series

Consider

$$\begin{aligned} X_t &= A\sin(2\pi\nu t) + B\cos(2\pi\nu t) \\ &= C\sin(2\pi\nu t + \phi), \end{aligned}$$

where $A, B$ are uncorrelated, mean zero, variance $\sigma^2 = 1$, and $C^2 = A^2 + B^2$, $\tan\phi = B/A$. Then

$$\begin{aligned} \mu_t = \mathrm{E}[X_t] &= 0 \\ \gamma(t, t+h) &= \cos(2\pi\nu h). \end{aligned}$$

So $\{X_t\}$ is stationary.

# An aside: Some trigonometric identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

$$\sin^2\theta + \cos^2\theta = 1,$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b,$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

# A periodic time series

For $X_t = A\sin(2\pi\nu t) + B\cos(2\pi\nu t)$, with uncorrelated $A, B$ (mean 0, variance $\sigma^2$), $\gamma(h) = \sigma^2\cos(2\pi\nu h)$.

The autocovariance of the sum of two uncorrelated time series is the sum of their autocovariances. Thus, the autocovariance of a sum of random sinusoids is a sum of sinusoids with the corresponding frequencies:

$$X_t = \sum_{j=1}^{k} \left(A_j \sin(2\pi\nu_j t) + B_j \cos(2\pi\nu_j t)\right),$$

$$\gamma(h) = \sum_{j=1}^{k} \sigma_j^2 \cos(2\pi\nu_j h),$$

where $A_j, B_j$ are uncorrelated, mean zero, and $\mathrm{Var}(A_j) = \mathrm{Var}(B_j) = \sigma_j^2$.

## A periodic time series

$$X_t = \sum_{j=1}^{k} \left(A_j \sin(2\pi\nu_j t) + B_j \cos(2\pi\nu_j t)\right), \quad \gamma(h) = \sum_{j=1}^{k} \sigma_j^2 \cos(2\pi\nu_j h).$$

Thus, we can represent $\gamma(h)$ using a Fourier series. The coefficients are the variances of the sinusoidal components.

The *spectral density* is the continuous analog: the Fourier transform of $\gamma$.

(The analogous *spectral representation* of a stationary process $X_t$ involves a *stochastic integral*—a sum of discrete components at a finite number of frequencies is a special case. We won't consider this representation in this course.)

## Spectral density

If a time series $\{X_t\}$ has autocovariance $\gamma$ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then we define its **spectral density** as

$$f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h}$$

for $-\infty < \nu < \infty$.

## Spectral density: Some facts

1. We have $\sum_{h=-\infty}^{\infty} \left|\gamma(h)e^{-2\pi i\nu h}\right| < \infty$.
   This is because $|e^{i\theta}| = |\cos\theta + i\sin\theta| = (\cos^2\theta + \sin^2\theta)^{1/2} = 1$, and because of the absolute summability of $\gamma$.

2. $f$ is periodic, with period $1$.
   This is true since $e^{-2\pi i\nu h}$ is a periodic function of $\nu$ with period $1$.

   Thus, we can restrict the domain of $f$ to $-1/2 \le \nu \le 1/2$. (The text does this.)

## Spectral density: Some facts

3. $f$ is even (that is, $f(\nu) = f(-\nu)$).

   To see this, write

$$
\begin{aligned}
f(\nu) &= \sum_{h=-\infty}^{-1} \gamma(h) e^{-2\pi i \nu h} + \gamma(0) + \sum_{h=1}^{\infty} \gamma(h) e^{-2\pi i \nu h}, \\
f(-\nu) &= \sum_{h=-\infty}^{-1} \gamma(h) e^{-2\pi i \nu (-h)} + \gamma(0) + \sum_{h=1}^{\infty} \gamma(h) e^{-2\pi i \nu (-h)}, \\
&= \sum_{h=1}^{\infty} \gamma(-h) e^{-2\pi i \nu h} + \gamma(0) + \sum_{h=-\infty}^{-1} \gamma(-h) e^{-2\pi i \nu h} \\
&= f(\nu).
\end{aligned}
$$

4. $f(\nu) \geq 0$.

## Spectral density: Some facts

5. $\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\nu h} f(\nu)\, d\nu.$

$$\begin{aligned}
\int_{-1/2}^{1/2} e^{2\pi i\nu h} f(\nu)\, d\nu &= \int_{-1/2}^{1/2} \sum_{j=-\infty}^{\infty} e^{-2\pi i\nu(j-h)} \gamma(j)\, d\nu \\
&= \sum_{j=-\infty}^{\infty} \gamma(j) \int_{-1/2}^{1/2} e^{-2\pi i\nu(j-h)}\, d\nu \\
&= \gamma(h) + \sum_{j\neq h} \frac{\gamma(j)}{2\pi i(j-h)} \left( e^{\pi i(j-h)} - e^{-\pi i(j-h)} \right) \\
&= \gamma(h) + \sum_{j\neq h} \frac{\gamma(j)\sin(\pi(j-h))}{\pi(j-h)} = \gamma(h).
\end{aligned}$$

# Example: White noise

For white noise $\{W_t\}$, we have seen that $\gamma(0) = \sigma_w^2$ and $\gamma(h) = 0$ for $h \neq 0$.

Thus,

$$\begin{aligned} f(\nu) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h} \\ &= \gamma(0) = \sigma_w^2. \end{aligned}$$

That is, the spectral density is constant across all frequencies: each frequency in the spectrum contributes equally to the variance. This is the origin of the name *white noise*: it is like white light, which is a uniform mixture of all frequencies in the visible spectrum.

# Example: AR(1)

For $X_t = \phi_1 X_{t-1} + W_t$, we have seen that $\gamma(h) = \sigma_w^2 \phi_1^{|h|}/(1-\phi_1^2)$. Thus,

$$
\begin{aligned}
f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h} &= \frac{\sigma_w^2}{1-\phi_1^2} \sum_{h=-\infty}^{\infty} \phi_1^{|h|} e^{-2\pi i \nu h} \\
&= \frac{\sigma_w^2}{1-\phi_1^2} \left( 1 + \sum_{h=1}^{\infty} \phi_1^h \left( e^{-2\pi i \nu h} + e^{2\pi i \nu h} \right) \right) \\
&= \frac{\sigma_w^2}{1-\phi_1^2} \left( 1 + \frac{\phi_1 e^{-2\pi i \nu}}{1-\phi_1 e^{-2\pi i \nu}} + \frac{\phi_1 e^{2\pi i \nu}}{1-\phi_1 e^{2\pi i \nu}} \right) \\
&= \frac{\sigma_w^2}{(1-\phi_1^2)} \frac{1 - \phi_1 e^{-2\pi i \nu} \phi_1 e^{2\pi i \nu}}{(1-\phi_1 e^{-2\pi i \nu})(1-\phi_1 e^{2\pi i \nu})} \\
&= \frac{\sigma_w^2}{1 - 2\phi_1 \cos(2\pi\nu) + \phi_1^2}.
\end{aligned}
$$

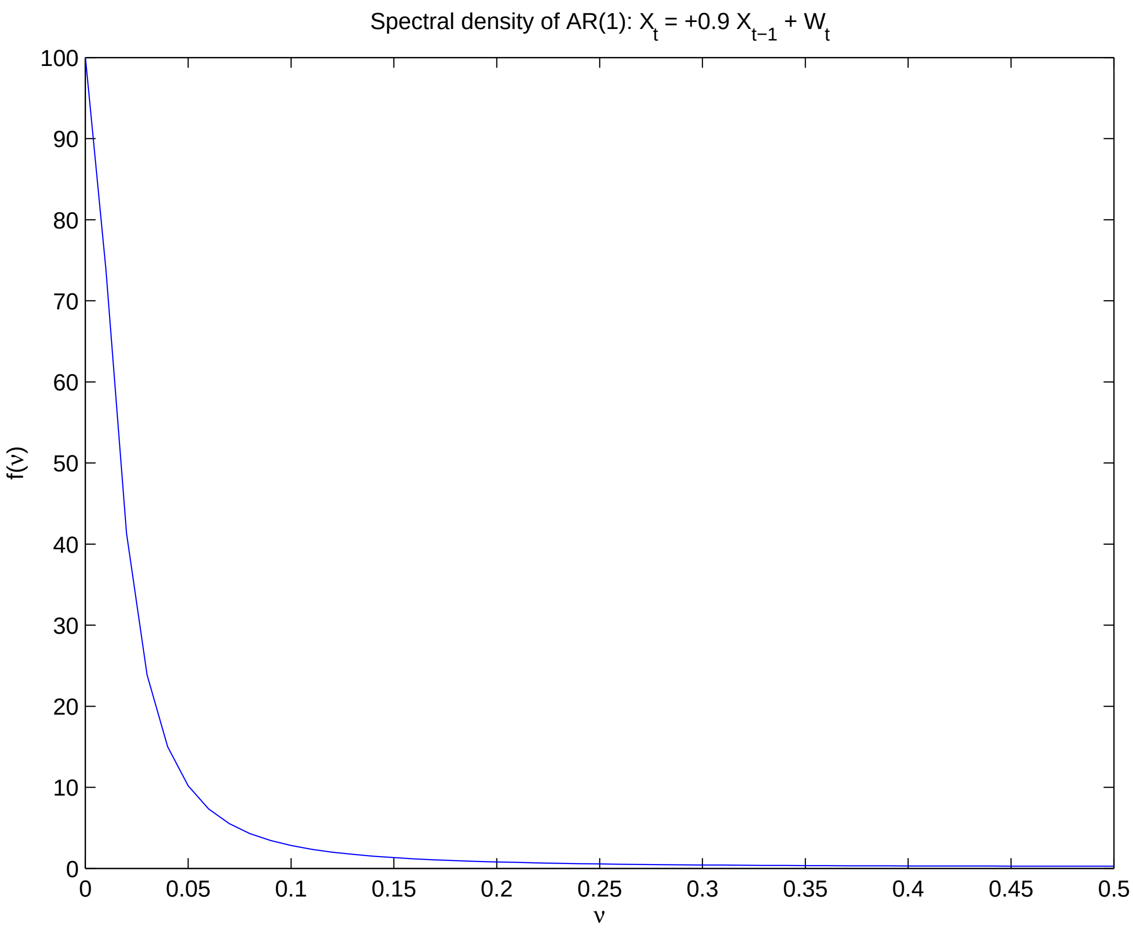
## Examples

White noise: $\{W_t\}$, $\gamma(0) = \sigma_w^2$ and $\gamma(h) = 0$ for $h \neq 0$.
$f(\nu) = \gamma(0) = \sigma_w^2$.

AR(1): $X_t = \phi_1 X_{t-1} + W_t$, $\gamma(h) = \sigma_w^2 \phi_1^{|h|}/(1 - \phi_1^2)$.
$f(\nu) = \frac{\sigma_w^2}{1-2\phi_1 \cos(2\pi\nu)+\phi_1^2}$.

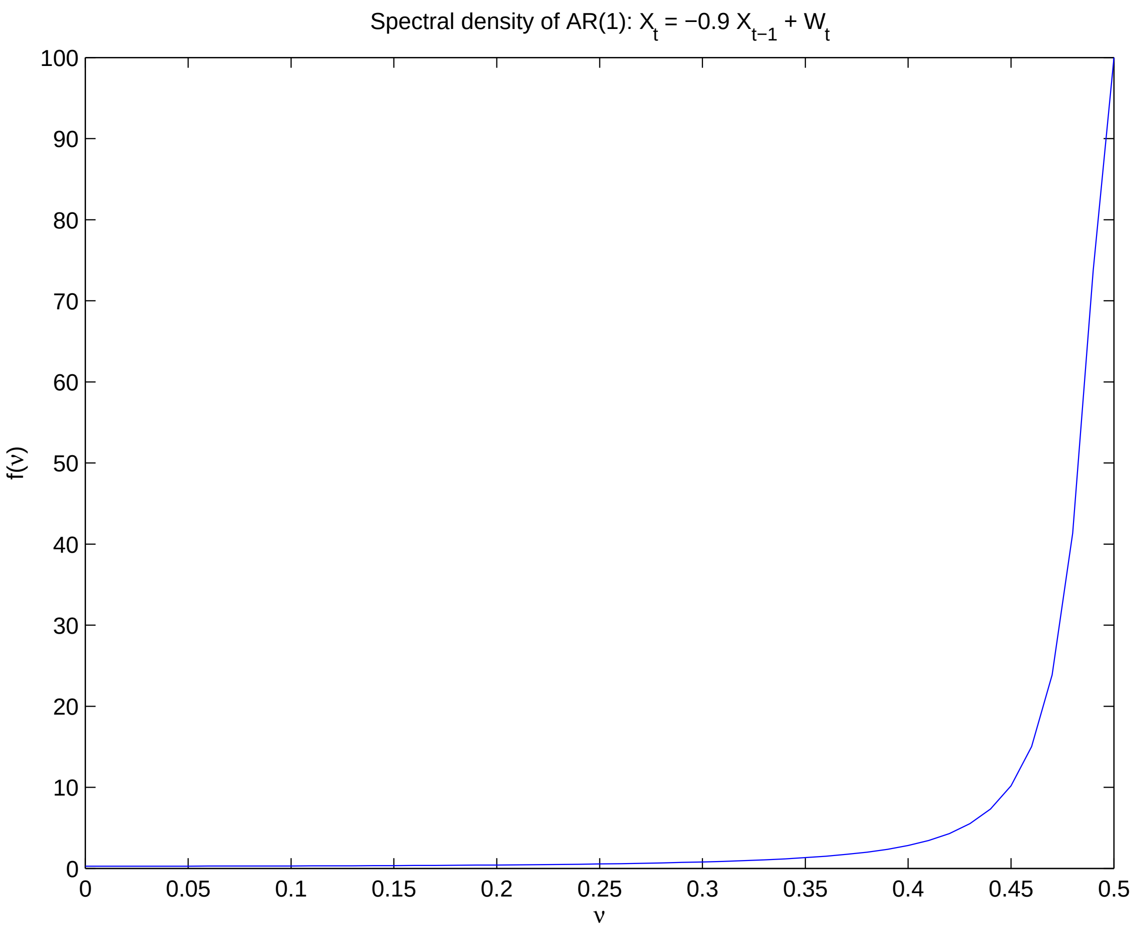
If $\phi_1 > 0$ (positive autocorrelation), spectrum is dominated by low frequency components—smooth in the time domain.
If $\phi_1 < 0$ (negative autocorrelation), spectrum is dominated by high frequency components—rough in the time domain.

# Example: AR(1)



Spectral density of AR(1): X_t = +0.9 X_{t−1} + W_t
f(ν)
ν
100
90
80
70
60
50
40
30
20
10
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

# Example: AR(1)



Spectral density of AR(1): X_t = −0.9 X_{t−1} + W_t
f(ν)
ν
0
10
20
30
40
50
60
70
80
90
100
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

# Example: MA(1)

$$X_t = W_t + \theta_1 W_{t-1}.$$

$$\gamma(h) = \begin{cases} \sigma_w^2(1+\theta_1^2) & \text{if } h = 0, \\ \sigma_w^2 \theta_1 & \text{if } |h| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} f(\nu) &= \sum_{h=-1}^{1} \gamma(h) e^{-2\pi i \nu h} \\ &= \gamma(0) + 2\gamma(1)\cos(2\pi\nu) \\ &= \sigma_w^2 \left(1 + \theta_1^2 + 2\theta_1 \cos(2\pi\nu)\right). \end{aligned}$$

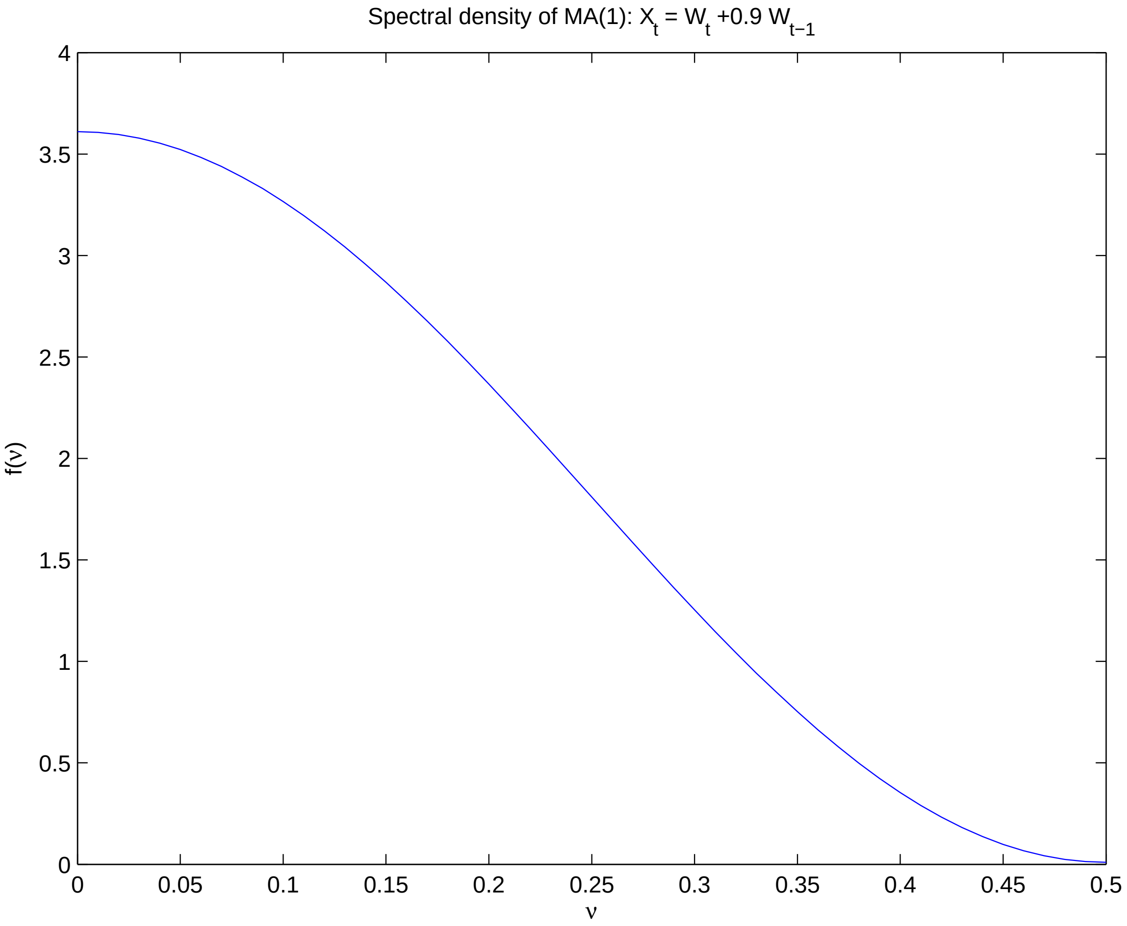
## Example: MA(1)

$$X_t = W_t + \theta_1 W_{t-1}.$$

$$f(\nu) = \sigma_w^2 \left(1 + \theta_1^2 + 2\theta_1 \cos(2\pi\nu)\right).$$

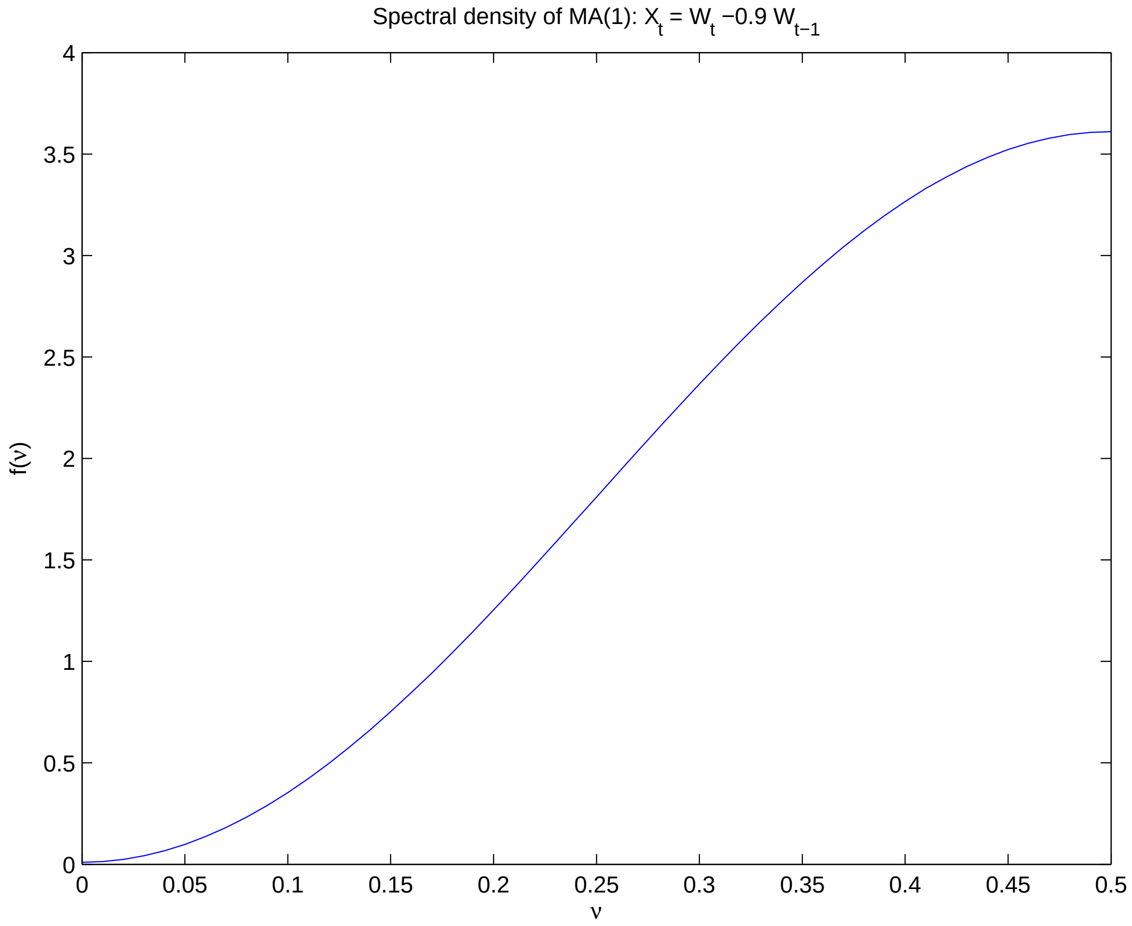
If $\theta_1 > 0$ (positive autocorrelation), spectrum is dominated by low frequency components—smooth in the time domain.
If $\theta_1 < 0$ (negative autocorrelation), spectrum is dominated by high frequency components—rough in the time domain.

# Example: MA(1)



Spectral density of MA(1): X_t = W_t +0.9 W_t−1
f(ν)
ν
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

# Example: MA(1)


Spectral density of MA(1): X_t = W_t −0.9 W_{t−1}
4
3.5
3
2.5
2
1.5
1
0.5
0
f(ν)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
ν

# Introduction to Time Series Analysis. Lecture 17.

1. Spectral density: Facts and examples.
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## Recall: A periodic time series

$$X_t = \sum_{j=1}^{k} \left(A_j \sin(2\pi\nu_j t) + B_j \cos(2\pi\nu_j t)\right)$$

$$= \sum_{j=1}^{k} (A_j^2 + B_j^2)^{1/2} \sin(2\pi\nu_j t + \tan^{-1}(B_j/A_j)).$$

$$\mathrm{E}[X_t] = 0$$

$$\gamma(h) = \sum_{j=1}^{k} \sigma_j^2 \cos(2\pi\nu_j h)$$

$$\sum_h |\gamma(h)| = \infty.$$

## Discrete spectral distribution function

For $X_t = A\sin(2\pi\lambda t) + B\cos(2\pi\lambda t)$, we have $\gamma(h) = \sigma^2\cos(2\pi\lambda h)$, and we can write

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\nu h}\,dF(\nu),$$

where $F$ is the discrete distribution

$$F(\nu) = \begin{cases} 0 & \text{if } \nu < -\lambda, \\ \frac{\sigma^2}{2} & \text{if } -\lambda \le \nu < \lambda, \\ \sigma^2 & \text{otherwise.} \end{cases}$$

## The spectral distribution function

For any stationary $\{X_t\}$ with autocovariance $\gamma$, we can write

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i \nu h} dF(\nu),$$

where $F$ is the *spectral distribution function* of $\{X_t\}$.

We can split $F$ into three components: discrete, continuous, and singular.

If $\gamma$ is absolutely summable, $F$ is continuous: $dF(\nu) = f(\nu)d\nu$.

If $\gamma$ is a sum of sinusoids, $F$ is discrete.

## The spectral distribution function

For $X_t = \sum_{j=1}^{k} (A_j \sin(2\pi\nu_j t) + B_j \cos(2\pi\nu_j t))$, the spectral distribution function is $F(\nu) = \sum_{j=1}^{k} \sigma_j^2 F_j(\nu)$, where

$$F_j(\nu) = \begin{cases} 0 & \text{if } \nu < -\nu_j, \\ \frac{1}{2} & \text{if } -\nu_j \leq \nu < \nu_j, \\ 1 & \text{otherwise.} \end{cases}$$

## Wold's decomposition

Notice that $X_t = \sum_{j=1}^{k} (A_j \sin(2\pi\nu_j t) + B_j \cos(2\pi\nu_j t))$ is deterministic (once we've seen the past, we can predict the future without error).

Wold showed that every stationary process can be represented as

$$X_t = X_t^{(d)} + X_t^{(n)},$$

where $X_t^{(d)}$ is purely deterministic and $X_t^{(n)}$ is purely nondeterministic. (c.f. the decomposition of a spectral distribution function as $F^{(d)} + F^{(c)}$.)

Example: $X_t = A \sin(2\pi\lambda t) + \frac{\theta(B)}{\phi(B)} W_t$.