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Uniform laws of large numbers: Bounding Rademacher complexity.

1. Growth function.

2. Vapnik-Chervonenkis dimension.
Recall: Uniform laws and Rademacher complexity

Definition: The Rademacher complexity of $F$ is $\mathbb{E}\|R_n\|_F$, where the empirical process $R_n$ is defined as

$$R_n(f) = \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(X_i) \right|,$$

where the $\epsilon_1, \ldots, \epsilon_n$ are Rademacher random variables: i.i.d. uniform on $\{\pm 1\}$. 

Recall: Uniform laws and Rademacher complexity

**Theorem:** For $F \subset [0, 1]^X$,

\[
\frac{1}{2} \mathbb{E}\|R_n\|_F - \sqrt{\frac{\log 2}{2n}} \leq \mathbb{E}\|P - P_n\|_F \leq 2\mathbb{E}\|R_n\|_F,
\]

and, with probability at least $1 - 2 \exp(-2\epsilon^2 n)$,

\[
\mathbb{E}\|P - P_n\|_F - \epsilon \leq \|P - P_n\|_F \leq \mathbb{E}\|P - P_n\|_F + \epsilon.
\]

Thus, $\mathbb{E}\|R_n\|_F \to 0$ iff $\|P - P_n\|_F \xrightarrow{a.s.} 0$. 
Lemma: [Finite Class Lemma] For $f \in F$ satisfying $|f(x)| \leq 1$,

$$
E\|R_n\|_F \leq E\sqrt{\frac{2 \log(|F(X_1^n) \cup -F(X_1^n)|)}{n}} \\
\leq \sqrt{\frac{2 \log(2E|F(X_1^n)|)}{n}}.
$$

[where $R_n$ is the Rademacher process:

$$R_n(f) = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(X_i).$$

and $F(X_1^n)$ is the set of restrictions of functions in $F$ to $X_1, \ldots, X_n$.]

Controlling Rademacher complexity: Growth function
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Proof: For $A \subseteq \mathbb{R}^n$ with $R = \max_{a \in A} \|a\|_2$, we saw that

$$
\mathbb{E} \sup_{a \in A} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i a_i \right| \leq \frac{R}{n} \sqrt{2 \log(|A \cup -A|)}.
$$

Here, we have $A = F(X_1^n)$, so $R \leq \sqrt{n}$, and we get

$$
\mathbb{E} \|R_n\|_F = \mathbb{E} \mathbb{E} \left[ \|R_n\|_{F(X_1^n)} | X_1, \ldots, X_n \right]
\leq \mathbb{E} \sqrt{\frac{2 \log(2|F(X_1^n)|)}{n}}
\leq \sqrt{\frac{2 \mathbb{E} \log(2|F(X_1^n)|)}{n}}
\leq \sqrt{\frac{2 \log(2\mathbb{E}|F(X_1^n)|)}{n}}.
$$
Controlling Rademacher complexity: Growth function

e.g. For the class of distribution functions, \( G = \{x \mapsto 1[x \leq \alpha] : \alpha \in \mathbb{R}\} \), we saw that \(|G(x^n_1)| \leq n + 1\). So \( \mathbb{E}\|R_n\|_F \leq \sqrt{\frac{2 \log 2(n+1)}{n}} \).

e.g. \( F \) parameterized by \( k \) bits:
If \( g \) maps to \([0, 1]\),
\[
F = \{x \mapsto g(x, \theta) : \theta \in \{0, 1\}^k\}
\]
\(|F(x^n_1)| \leq 2^k\),
\[
\mathbb{E}\|R_n\|_F \leq \sqrt{\frac{2(k + 1) \log 2}{n}}.
\]
Notice that \( \mathbb{E}\|R_n\|_F \to 0 \).
**Growth function**

**Definition:** For a class \( F \subseteq \{0, 1\}^X \), the growth function is

\[
\Pi_F(n) = \max\{|F(x_1^n)| : x_1, \ldots, x_n \in X\}.
\]

- \( \mathbb{E}\|R_n\|_F \leq \sqrt{\frac{2\log(2\Pi_F(n))}{n}} \).
- \( \Pi_F(n) \leq |F|, \lim_{n \to \infty} \Pi_F(n) = |F| \).
- \( \Pi_F(n) \leq 2^n \). (But then this gives no useful bound on \( \mathbb{E}\|R_n\|_F \).)
- Notice that \( \log \Pi_F(n) = o(n) \) implies \( \mathbb{E}\|R_n\|_F \to 0 \).
Vapnik-Chervonenkis dimension

Definition: A class $F \subseteq \{0, 1\}^\mathcal{X}$ shatters $\{x_1, \ldots, x_d\} \subseteq \mathcal{X}$ means that $|F(x_1^d)| = 2^d$.

The Vapnik-Chervonenkis dimension of $F$ is

$$d_{VC}(F) = \max \{d : \text{some } x_1, \ldots, x_d \in \mathcal{X} \text{ is shattered by } F\}$$

$$= \max \{d : \Pi_F(d) = 2^d\}.$$
**Theorem:** [Vapnik-Chervonenkis] $d_{VC}(F) \leq d$ implies

$$\Pi_F(n) \leq \sum_{i=0}^{d} \binom{n}{i}.$$ 

If $n \geq d$, the latter sum is no more than $(\frac{en}{d})^d$.

So the VC-dimension is a single integer summary of the growth function: either it is finite, and $\Pi_F(n) = O(n^d)$, or $\Pi_F(n) = 2^n$. No other growth is possible.

$$\Pi_F(n) \begin{cases} = 2^n & \text{if } n \leq d, \\ \leq (e/d)^d n^d & \text{if } n > d. \end{cases}$$
Thus, for $d_{VC}(F) \leq d$ and $n \geq d$, we have

$$E\|R_n\|_F \leq \sqrt{\frac{2\log(2\Pi_F(n))}{n}} \leq \sqrt{\frac{2\log 2 + 2d \log(en/d)}{n}}.$$
Vapnik-Chervonenkis dimension: Examples

e.g.: $F = \{x \mapsto 1[x \leq \alpha] : \alpha \in \mathbb{R}\}$.
$d_{VC}(F) = 1$.

e.g.: $F = \{x \mapsto 1[x \text{ below and to left of } y] : y \in \mathbb{R}^2\}$.
$d_{VC}(F) = 2$. [PICTURE]

e.g.: $F = \{x \mapsto 1[x \in H] : H \text{ halfspace}\}$.
For $d = 2$, $d_{VC}(F) = 3$. [PICTURE]
Thresholded linear functions:

\[ F = \{ x \mapsto 1[g(x) \geq 0] : g \in G \}, \quad \text{where } G \text{ is a linear space.} \]

Then \( d_{VC}(F) = \text{dim}(G) \).

Let \( d = \text{dim}(G) \). To see that \( d_{VC}(F) \geq d \), suppose that \( g_1, \ldots, g_d \in G \) is a set of linearly independent functions. Then a fundamental result of linear algebra (row rank=column rank) implies that there are \( d \) points \( x_1, \ldots, x_d \) such that the vectors \( g_1(x_1^d), \ldots, g_d(x_1^d) \) are linearly independent. Let \( M \) be the \( d \times d \) matrix of these values. Since \( G \) is linear, any linear combination of these functions is also in \( G \). For coefficients \( v \), this function’s value on these \( d \) points is given by \( Mv \). Since \( M \) is full rank, for any \( y \), we can find a \( v \) so that \( Mv = y \). In particular, \( y \) can have any sequence of signs, so \( x_1, \ldots, x_d \) are shattered by \( G \).
To see that $d_{VC}(F) \leq d$, consider any $x_1, \ldots, x_{d+1}$. Then

$$\{(g(x_1), \ldots, g(x_{d+1})) : g \in G\}$$

is a linear subspace of dimension $d$. So there must be a non-zero $v \in \mathbb{R}^{d+1}$ for which $\sum_i v_i g(x_i) = 0$ for all $g \in G$. Suppose that $G$ shatters this set of $d+1$ points. Wlog, suppose some $v_i > 0$. Consider a $g$ for which $g(x_i) < 0$ for exactly those $i$ with $v_i > 0$. Then

$$0 = \sum_i v_i g(x_i) = \sum_{i:v_i \leq 0} v_i g(x_i) + \sum_{i:v_i > 0} v_i g(x_i) \leq 0 + \sum_{i:v_i > 0} v_i g(x_i) < 0,$$

which is a contradiction.
Vapnik-Chervonenkis Lemma: Proof

Fix $x_1, \ldots, x_n$ and consider the table of values of $F(x_1^n)$:

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$f_2$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$f_3$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$f_4$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The cardinality of $F(x_1^n)$ is the number of distinct rows.
Vapnik-Chervonenkis Lemma: Proof

Consider the following shifting transformation of the table: For a column $i$, change each 1 to a 0, unless it would lead to a row that is already in the table.

Shifting the columns from left to right gives:

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$f_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$f_3$</td>
<td>0</td>
<td>0</td>
<td>1</td>
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</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Vapnik-Chervonenkis Lemma: Proof

Suppose this shifting operation is performed column-by-column until it leads to no change of the table. Then:

- The number of rows does not change.
- Consider a row with any 1s. Every row with some of those 1s changed to 0s is in the table.
Vapnik-Chervonenkis Lemma: Proof

- The VC-dimension never increases. (Consider a set that is shattered after shifting a column. If the set does not include the column, it was certainly shattered before shifting. If it does include the column, we need to show that the set was shattered before. Suppose that an entry was shifted down to a zero. The 1s that remain in the column are there because there was a row before shifting that is identical but for a 0 in that column. So the newly shifted 0 plays no role in the shattering.)

- So no row has more than $d$ 1s.
Thus, the number of rows is no more than $\sum_{i=0}^{d} \binom{n}{i}$.

Finally, for $n \geq d$,

$$\sum_{i=0}^{d} \binom{n}{i} \leq \left( \frac{n}{d} \right)^d \sum_{i=0}^{d} \binom{n}{i} \left( \frac{d}{n} \right)^i$$

$$= \left( \frac{n}{d} \right)^d \left( 1 + \frac{d}{n} \right)^n$$  (binomial thm)

$$\leq \left( \frac{en}{d} \right)^d.$$
Consider a parameterized class of binary-valued functions,

\[ F = \{ x \mapsto f(x, \theta) : \theta \in \mathbb{R}^p \} , \]

where \( f : \mathbb{R}^m \times \mathbb{R}^p \to \{ \pm 1 \} \).

Suppose that \( f \) can be computed using no more than \( t \) operations of the following kinds:

1. arithmetic (+, −, ×, /),
2. comparisons (>, =, <),
3. output ±1.

**Theorem:** \( d_{VC}(F) \leq 4p(t + 2) \).