Introduction to Time Series Analysis. Lecture 5.

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Last lecture:

1. ACF, sample ACF
2. Properties of the sample ACF
3. Convergence in mean square
Introduction to Time Series Analysis. Lecture 5.
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1. AR(1) as a linear process
2. Causality
3. Invertibility
4. AR(p) models
5. ARMA(p,q) models
Let \( \{X_t\} \) be the stationary solution to \( X_t - \phi X_{t-1} = W_t \), where \( W_t \sim WN(0, \sigma^2) \).

If \( |\phi| < 1 \),

\[
X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}
\]

is the unique solution:

- This infinite sum converges in mean square, since \( |\phi| < 1 \) implies \( \sum |\phi^j| < \infty \).
- It satisfies the AR(1) recurrence.
AR(1) in terms of the back-shift operator

We can write

\[ X_t - \phi X_{t-1} = W_t \]

\[ \iff \]

\[ (1 - \phi B) X_t = W_t \]

\[ \iff \]

\[ \phi(B) X_t = W_t \]

Recall that \( B \) is the back-shift operator: \( BX_t = X_{t-1} \).
AR(1) in terms of the back-shift operator

Also, we can write

\[ X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j} \]

⇔

\[ X_t = \sum_{j=0}^{\infty} \phi^j B^j W_t \]

⇔

\[ X_t = \pi(B)W_t \]
AR(1) in terms of the back-shift operator

With these definitions:

\[ \pi(B) = \sum_{j=0}^{\infty} \phi^j B^j \quad \text{and} \quad \phi(B) = 1 - \phi B, \]

we can check that \( \pi(B) = \phi(B)^{-1} \):

\[ \pi(B)\phi(B) = \sum_{j=0}^{\infty} \phi^j B^j (1 - \phi B) = \sum_{j=0}^{\infty} \phi^j B^j - \sum_{j=1}^{\infty} \phi^j B^j = 1. \]

Thus,

\[ \phi(B)X_t = W_t \]

\[ \Rightarrow \quad \pi(B)\phi(B)X_t = \pi(B)W_t \]

\[ \Leftrightarrow \quad X_t = \pi(B)W_t. \]
AR(1) in terms of the back-shift operator

Notice that manipulating operators like $\phi(B)$, $\pi(B)$ is like manipulating polynomials:

$$\frac{1}{1 - \phi z} = 1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \ldots,$$

provided $|\phi| < 1$ and $|z| \leq 1$. 

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**AR(1) and Causality**

Let $X_t$ be the stationary solution to

$$X_t - \phi X_{t-1} = W_t,$$

where $W_t \sim \mathcal{WN}(0, \sigma^2)$.

If $|\phi| < 1$,

$$X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}.$$
AR(1) and Causality

If $|\phi| > 1$, $\pi(B)W_t$ does not converge. But we can rearrange

$$X_t = \phi X_{t-1} + W_t$$

as

$$X_{t-1} = \frac{1}{\phi} X_t - \frac{1}{\phi} W_t,$$

and we can check that the unique stationary solution is

$$X_t = - \sum_{j=1}^{\infty} \phi^{-j} W_{t+j}.$$ 

But... $X_t$ depends on future values of $W_t$. 
A linear process \( \{X_t\} \) is **causal** (strictly, a **causal function** of \( \{W_t\} \)) if there is a

\[
\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots
\]

with

\[
\sum_{j=0}^{\infty} |\psi_j| < \infty
\]

and

\[X_t = \psi(B)W_t.\]
Causality is a property of \( \{X_t\} \) and \( \{W_t\} \).

Consider the AR(1) process defined by \( \phi(B)X_t = W_t \) (with \( \phi(B) = 1 - \phi B \)):

\[
\phi(B)X_t = W_t \quad \text{is causal}
\]

iff \( |\phi| < 1 \)

iff the root \( z_1 \) of the polynomial \( \phi(z) = 1 - \phi z \) satisfies \( |z_1| > 1 \).
AR(1) and Causality

- Consider the AR(1) process $\phi(B)X_t = W_t$ (with $\phi(B) = 1 - \phi B$):
  If $|\phi| > 1$, we can define an equivalent causal model,

  $$X_t - \phi^{-1}X_{t-1} = \tilde{W}_t,$$

  where $\tilde{W}_t$ is a new white noise sequence.
AR(1) and Causality

- Is an MA(1) process causal?
Introduction to Time Series Analysis. Lecture 5.

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Define

\[ X_t = W_t + \theta W_{t-1} \]

\[ = (1 + \theta B)W_t. \]

If \(|\theta| < 1\), we can write

\[ (1 + \theta B)^{-1}X_t = W_t \]

\[ \iff (1 - \theta B + \theta^2 B^2 - \theta^3 B^3 + \cdots)X_t = W_t \]

\[ \iff \sum_{j=0}^{\infty} (-\theta)^j X_{t-j} = W_t. \]

That is, we can write \( W_t \) as a causal function of \( X_t \).
We say that this MA(1) is invertible.
\[ X_t = W_t + \theta W_{t-1} \]

If \(|\theta| > 1\), the sum \(\sum_{j=0}^{\infty} (-\theta)^j X_{t-j}\) diverges, but we can write
\[ W_{t-1} = -\theta^{-1} W_t + \theta^{-1} X_t. \]

Just like the noncausal AR(1), we can show that
\[ W_t = -\sum_{j=1}^{\infty} (-\theta)^{-j} X_{t+j}. \]

That is, we can write \(W_t\) as a linear function of \(X_t\), but it is not causal. We say that this MA(1) is not invertible.
A linear process \( \{X_t\} \) is **invertible** (strictly, an invertible function of \( \{W_t\} \)) if there is a

\[
\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots
\]

with

\[
\sum_{j=0}^{\infty} |\pi_j| < \infty
\]

and

\[W_t = \pi(B) X_t.\]
MA(1) and Invertibility

- Invertibility is a property of \( \{X_t\} \) and \( \{W_t\} \).
- Consider the MA(1) process defined by \( X_t = \theta(B)W_t \) (with \( \theta(B) = 1 + \theta B \)):
  \[
  X_t = \theta(B)W_t \quad \text{is invertible}
  \]
  iff  \( |\theta| < 1 \)
  iff  the root \( z_1 \) of the polynomial \( \theta(z) = 1 + \theta z \) satisfies \( |z_1| > 1 \).
Consider the MA(1) process $X_t = \theta(B)W_t$ (with $\theta(B) = 1 + \theta B$): If $|\theta| > 1$, we can define an equivalent invertible model in terms of a new white noise sequence.

Is an AR(1) process invertible?
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AR(p): Autoregressive models of order $p$

An AR(p) process $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t,$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

Equivalently, $\phi(B) X_t = W_t,$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$. 
Recall: For $p = 1$ (AR(1)), $\phi(B) = 1 - \phi_1 B$.

This is an AR(1) model only if there is a stationary solution to $\phi(B)X_t = W_t$, which is equivalent to $|\phi_1| \neq 1$.

This is equivalent to the following condition on $\phi(z) = 1 - \phi_1 z$:

$$\forall z \in \mathbb{R}, \phi(z) = 0 \Rightarrow z \neq \pm 1$$

equivalently, $\forall z \in \mathbb{C}, \phi(z) = 0 \Rightarrow |z| \neq 1$,

where $\mathbb{C}$ is the set of complex numbers.
AR(p): Constraints on $\phi$

Stationarity: $\forall z \in \mathbb{C}, \phi(z) = 0 \Rightarrow |z| \neq 1,$

where $\mathbb{C}$ is the set of complex numbers.

$\phi(z) = 1 - \phi_1 z$ has one root at $z_1 = 1/\phi_1 \in \mathbb{R}$.

But the roots of a degree $p > 1$ polynomial might be complex.

For stationarity, we want the roots of $\phi(z)$ to avoid the \textbf{unit circle},

$\{z \in \mathbb{C} : |z| = 1\}.$
AR(p): Stationarity and causality

Theorem: A (unique) stationary solution to $\phi(B)X_t = W_t$ exists iff
\[ \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0 \Rightarrow |z| \neq 1. \]

This AR(p) process is causal iff
\[ \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = 0 \Rightarrow |z| > 1. \]
Recall: Causality

A linear process \( \{X_t\} \) is causal (strictly, a causal function of \( \{W_t\} \)) if there is a

\[
\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots
\]

with

\[
\sum_{j=0}^{\infty} |\psi_j| < \infty
\]

and

\[
X_t = \psi(B)W_t.
\]
AR(p): Roots outside the unit circle implies causal (Details)

\[ \forall z \in \mathbb{C}, \ |z| \leq 1 \Rightarrow \phi(z) \neq 0 \]

\[ \Leftrightarrow \exists \{\psi_j\}, \delta > 0, \ \forall |z| \leq 1 + \delta, \ \frac{1}{\phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j. \]

\[ \Rightarrow \forall |z| \leq 1 + \delta, \ |\psi_j z^j| \rightarrow 0, \ \left( |\psi_j|^{1/j} |z| \right)^j \rightarrow 0 \]

\[ \Rightarrow \exists j_0, \ \forall j \geq j_0, \ |\psi_j|^{1/j} \leq \frac{1}{1 + \delta/2} \Rightarrow \sum_{j=0}^{\infty} |\psi_j| < \infty. \]

So if \( |z| \leq 1 \Rightarrow \phi(z) \neq 0 \), then \( S_m = \sum_{j=0}^{m} \psi_j B^j W_t \) converges in mean square, so we have a stationary, causal time series \( X_t = \phi^{-1}(B)W_t \).
Calculating $\psi$ for an AR(p): matching coefficients

Example: $X_t = \psi(B)W_t \iff (1 - 0.5B + 0.6B^2)X_t = W_t,$

so $1 = \psi(B)(1 - 0.5B + 0.6B^2)$

$\iff 1 = (\psi_0 + \psi_1B + \psi_2B^2 + \cdots)(1 - 0.5B + 0.6B^2)$

$\iff 1 = \psi_0,$

$0 = \psi_1 - 0.5\psi_0,$

$0 = \psi_2 - 0.5\psi_1 + 0.6\psi_0,$

$0 = \psi_3 - 0.5\psi_2 + 0.6\psi_1,$

$\vdots$
Calculating $\psi$ for an AR(p): example

\[
\begin{align*}
\Leftrightarrow & \quad 1 = \psi_0, \quad 0 = \psi_j \quad (j \leq 0), \\
& \quad 0 = \psi_j - 0.5\psi_{j-1} + 0.6\psi_{j-2} \\
\Leftrightarrow & \quad 1 = \psi_0, \quad 0 = \psi_j \quad (j \leq 0), \\
& \quad 0 = \phi(B)\psi_j.
\end{align*}
\]

We can solve these linear difference equations in several ways:

- numerically, or
- by guessing the form of a solution and using an inductive proof, or
- by using the theory of linear difference equations.
Calculating $\psi$ for an AR(p): general case

$$\phi(B)X_t = W_t, \quad \Leftrightarrow \quad X_t = \psi(B)W_t$$

so \[ 1 = \psi(B)\phi(B) \]

$\Leftrightarrow$ \[ 1 = (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p) \]

$\Leftrightarrow$ \[ 1 = \psi_0, \]

\[ 0 = \psi_1 - \phi_1 \psi_0, \]

\[ 0 = \psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0, \]

\[ \vdots \]

$\Leftrightarrow$ \[ 1 = \psi_0, \quad 0 = \psi_j \quad (j < 0), \]

\[ 0 = \phi(B)\psi_j. \]
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An ARMA(p,q) process \( \{ X_t \} \) is a stationary process that satisfies
\[
X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q},
\]
where \( \{ W_t \} \sim WN(0, \sigma^2) \).

- AR(p) = ARMA(p,0): \( \theta(B) = 1 \).
- MA(q) = ARMA(0,q): \( \phi(B) = 1 \).
ARMA processes

Can accurately approximate many stationary processes:

For any stationary process with autocovariance $\gamma$, and any $k > 0$, there is an ARMA process $\{X_t\}$ for which

$$\gamma_X(h) = \gamma(h), \quad h = 0, 1, \ldots, k.$$
**ARMA(p,q): Autoregressive moving average models**

An ARMA(p,q) process \( \{X_t\} \) is a stationary process that satisfies
\[
X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q},
\]
where \( \{W_t\} \sim WN(0, \sigma^2) \).

Usually, we insist that \( \phi_p, \theta_q \neq 0 \) and that the polynomials
\[
\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p, \quad \theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q
\]
have no common factors. This implies it is not a lower order ARMA model.