Introduction to Time Series Analysis. Lecture 2.

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Last lecture:

1. Objectives of time series analysis.
2. Time series models.
Introduction to Time Series Analysis. Lecture 2.
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1. Stationarity
2. Autocovariance, autocorrelation
3. MA, AR, linear processes
4. Sample autocorrelation function
Stationarity

\{X_t\} is strictly stationary if
for all \(k, t_1, \ldots, t_k, x_1, \ldots, x_k,\) and \(h,\)

\[ P(X_{t_1} \leq x_1, \ldots, X_{t_k} \leq x_k) = P(X_{t_1+h} \leq x_1, \ldots, X_{t_k+h} \leq x_k) \]

i.e., shifting the time axis does not affect the distribution.

We shall consider second-order properties only.
Suppose that \( \{X_t\} \) is a time series with \( \mathbb{E}[X_t^2] < \infty \).

Its \textbf{mean function} is

\[
\mu_t = \mathbb{E}[X_t].
\]

Its \textbf{autocovariance function} is

\[
\gamma_X(s, t) = \text{Cov}(X_s, X_t) \\
= \mathbb{E}[(X_s - \mu_s)(X_t - \mu_t)].
\]
Weak Stationarity

We say that \( \{X_t\} \) is (weakly) stationary if

1. \( \mu_t \) is independent of \( t \), and
2. For each \( h \), \( \gamma_X(t + h, t) \) is independent of \( t \).

In that case, we write

\[ \gamma_X(h) = \gamma_X(h, 0). \]
The autocorrelation function (ACF) of \( \{X_t\} \) is defined as

\[
\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \frac{\text{Cov}(X_{t+h}, X_t)}{\text{Cov}(X_t, X_t)} = \text{Corr}(X_{t+h}, X_t).
\]
**Stationarity**

**Example:** i.i.d. noise, $\mathbb{E}[X_t] = 0$, $\mathbb{E}[X_t^2] = \sigma^2$. We have

$$
\gamma_X(t + h, t) = \begin{cases} 
\sigma^2 & \text{if } h = 0, \\
0 & \text{otherwise}.
\end{cases}
$$

Thus,

1. $\mu_t = 0$ is independent of $t$.
2. $\gamma_X(t + h, t) = \gamma_X(h, 0)$ for all $t$.

So $\{X_t\}$ is stationary.

Similarly for any white noise (uncorrelated, zero mean), $X_t \sim WN(0, \sigma^2)$. 
Stationarity

**Example:** Random walk, $S_t = \sum_{i=1}^{t} X_i$ for i.i.d., mean zero $\{X_t\}$. We have $E[S_t] = 0$, $E[S_t^2] = t\sigma^2$, and

$$\gamma_S(t + h, t) = \text{Cov}(S_{t+h}, S_t)$$

$$= \text{Cov} \left( S_t + \sum_{s=1}^{h} X_{t+s}, S_t \right)$$

$$= \text{Cov}(S_t, S_t) = t\sigma^2.$$

1. $\mu_t = 0$ is independent of $t$, but
2. $\gamma_S(t + h, t)$ is not.

So $\{S_t\}$ is not stationary.
An aside: covariances

\[ \text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z), \]
\[ \text{Cov}(aX, Y) = a \text{Cov}(X, Y), \]

Also if \( X \) and \( Y \) are independent (e.g., \( X = c \)), then

\[ \text{Cov}(X, Y) = 0. \]
Stationarity

Example: MA(1) process (Moving Average):

\[ X_t = W_t + \theta W_{t-1}, \quad \{W_t\} \sim WN(0, \sigma^2). \]

We have \( E[X_t] = 0 \), and

\[
\gamma_X(t + h, t) = E(X_{t+h}X_t) \\
= E[(W_{t+h} + \theta W_{t+h-1})(W_t + \theta W_{t-1})] \\
= \begin{cases} 
\sigma^2(1 + \theta^2) & \text{if } h = 0, \\
\sigma^2\theta & \text{if } h = \pm1, \\
0 & \text{otherwise.}
\end{cases}
\]

Thus, \( \{X_t\} \) is stationary.
ACF of the MA(1) process

MA(1): $X_t = Z_t + \theta Z_{t-1}$

$\theta/(1+\theta^2)$
**Stationarity**

**Example:** AR(1) process (AutoRegressive):

\[ X_t = \phi X_{t-1} + W_t, \quad \{W_t\} \sim WN(0, \sigma^2). \]

Assume that \( X_t \) is stationary and \(|\phi| < 1\). Then we have

\[
\begin{align*}
E[X_t] &= \phi E[X_{t-1}] \\
&= 0 \quad \text{(from stationarity)} \\
E[X_t^2] &= \phi^2 E[X_{t-1}^2] + \sigma^2 \\
&= \frac{\sigma^2}{1 - \phi^2} \quad \text{(from stationarity),}
\end{align*}
\]


**Stationarity**

**Example:** AR(1) process, \( X_t = \phi X_{t-1} + W_t, \quad \{W_t\} \sim WN(0, \sigma^2) \).

Assume that \( X_t \) is stationary and \(|\phi| < 1\). Then we have

\[
\begin{align*}
E[X_t] &= 0, \quad E[X_t^2] = \frac{\sigma^2}{1 - \phi^2} \\
\gamma_X(h) &= \text{Cov}(\phi X_{t+h-1} + W_{t+h}, X_t) \\
&= \phi \text{Cov}(X_{t+h-1}, X_t) \\
&= \phi \gamma_X(h - 1) \\
&= \phi^{|h|} \gamma_X(0) \quad \text{(check for } h > 0 \text{ and } h < 0) \\
&= \frac{\phi^{|h|} \sigma^2}{1 - \phi^2}.
\end{align*}
\]
ACF of the AR(1) process

AR(1): $X_t = \phi X_{t-1} + Z_t$
Linear Processes

An important class of stationary time series:

\[ X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j} \]

where \( \{W_t\} \sim WN(0, \sigma_w^2) \) and \( \mu, \psi_j \) are parameters satisfying

\[ \sum_{j=-\infty}^{\infty} |\psi_j| < \infty. \]
Linear Processes

\[ X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j} \]

We have

\[ \mu_X = \mu \]

\[ \gamma_X(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{h+j}. \] (why?)
Examples of Linear Processes: White noise

\[ X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j} \]

Choose \( \mu \),

\[ \psi_j = \begin{cases} 
1 & \text{if } j = 0, \\
0 & \text{otherwise.} 
\end{cases} \]

Then \( \{X_t\} \sim WN(\mu, \sigma^2_W) \). (why?)
Examples of Linear Processes: MA(1)

\[ X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j} \]

Choose \( \mu = 0 \)

\[ \psi_j = \begin{cases} 1 & \text{if } j = 0, \\ \theta & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases} \]

Then \( X_t = W_t + \theta W_{t-1} \). (why?)
Examples of Linear Processes: AR(1)

\[ X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j} \]

Choose \( \mu = 0 \)

\[ \psi_j = \begin{cases} 
\phi^j & \text{if } j \geq 0, \\
0 & \text{otherwise.} 
\end{cases} \]

Then for \( |\phi| < 1 \), we have \( X_t = \phi X_{t-1} + W_t \).
Recall: Suppose that \( \{X_t\} \) is a stationary time series. Its **mean** is
\[
\mu = \mathbb{E}[X_t].
\]
Its **autocovariance function** is
\[
\gamma(h) = \text{Cov}(X_{t+h}, X_t) = \mathbb{E}[(X_{t+h} - \mu)(X_t - \mu)].
\]
Its **autocorrelation function** is
\[
\rho(h) = \frac{\gamma(h)}{\gamma(0)}.
\]
Estimating the ACF: Sample ACF

For observations $x_1, \ldots, x_n$ of a time series, the **sample mean** is

$$\bar{x} = \frac{1}{n} \sum_{t=1}^{n} x_t.$$

The **sample autocovariance function** is

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \bar{x})(x_t - \bar{x}), \quad \text{for } -n < h < n.$$

The **sample autocorrelation function** is

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}.$$
Sample autocovariance function:

\[ \hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \bar{x})(x_t - \bar{x}). \]

\[ \approx \text{the sample covariance of } (x_1, x_{h+1}), \ldots, (x_{n-h}, x_n) \text{, except that} \]

- we normalize by \( n \) instead of \( n - h \), and
- we subtract the full sample mean.
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