Homework 1 solutions

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1. To check that $\{X_t\}$ is white noise, we need to compute its means and covariances. For the means, $\mathbb{E}X_t = \mathbb{E}W_t(1 - W_{t-1})Z_t = (\mathbb{E}W_t)(1 - \mathbb{E}W_{t-1})(\mathbb{E}Z_t) = 0$. For the covariances,

$$\begin{aligned}\gamma(s,t) &= \mathbb{E}\big(W_s(1 - W_{s-1})Z_s W_t(1 - W_{t-1})Z_t\big) \\ &= \mathbb{E}\big(W_s(1 - W_{s-1})W_t(1 - W_{t-1})\big) \cdot \mathbb{E}Z_s Z_t.\end{aligned}$$

If $s \neq t$ then the last term is $\mathbb{E}Z_s Z_t = \mathbb{E}Z_s \cdot \mathbb{E}Z_t = 0$. Therefore $\{X_t\}$ is uncorrelated. If $s = t$ then $\mathbb{E}Z_s Z_t = \mathbb{E}Z_t^2 = 1$ and so

$$\gamma(t,t) = \mathbb{E}W_t^2(1 - W_{t-1})^2 = \frac{1}{4}.$$

Thus, $\{X_t\}$ has constant variance. Hence it is white noise.

To show that $\{X_t\}$ is not i.i.d, note that $X_{t-1} = 1$ implies that $W_{t-1} = 1$, which implies that $X_t = 0$. Therefore

$$P(X_{t-1} = 1, X_t = 1) = 0.$$

Since this is not equal to $P(X_{t-1} = 1)P(X_t = 1) = 1/64$, X_t and X_{t-1} are not independent.

2. (a) $X_t = W_t - W_{t-3}$ is a stationary process: $\mathbb{E}X_t = \mathbb{E}W_t - \mathbb{E}W_{t-3} = 0$ and

$$\begin{aligned}\gamma(s,t) = \mathbb{E}X_s X_t &= \mathbb{E}W_s W_t + \mathbb{E}W_s W_{t-3} + \mathbb{E}W_{s-3}W_t + \mathbb{E}W_{s-3}W_{t-3} \\ &= \mathbb{1}_{\{s=t\}} + \cdot \mathbb{1}_{\{s=t-3\}} + \cdot \mathbb{1}_{\{s-3=t\}} + \cdot \mathbb{1}_{\{s-3=t-3\}} \\ &= 2 \cdot \mathbb{1}_{\{|s-t|=0\}} + \mathbb{1}_{\{|s-t|=3\}},\end{aligned}$$

which is a function of $|s - t|$.

(b) $X_t = W_3$ is a stationary process because $\mathbb{E}X_t = \mathbb{E}W_3 = 0$ and $\mathbb{E}X_s X_t = \mathbb{E}W_3^2 = 1$.

(c) $X_t = W_3 + t$ is not a stationary process because its mean is not constant: $\mathbb{E}X_t = t$.

(d) $X_t = W_t^2$ is a stationary process: $\mathbb{E}X_t = \mathbb{E}W_t^2 = 1$ and

$$\mathbb{E}X_s X_t = \mathbb{E}W_s^2 W_t^2 = \begin{cases} 3 & \text{if } s = t \\ 1 & \text{if } s \neq t. \end{cases}$$

(e) $X_t = W_t W_{t-2}$ is a stationary process: $\mathbb{E}X_t = \mathbb{E}W_t \mathbb{E}W_{t-2} = 0$ and $\gamma(s,t) = \mathbb{E}W_s W_{s-2} W_t W_{t-2} = \mathbb{1}_{\{s=t\}}$.

3. If $X_t = W_{t-1} + 2W_t + W_{t+1}$, then

$$\begin{aligned}
\gamma(t,t) &= \mathbb{E}(W_{t-1} + 2W_t + W_{t+1})^2 = \mathbb{E}W_{t-1}^2 + 4\mathbb{E}W_t^2 + \mathbb{E}W_{t+1}^2 = 6\sigma_w^2 \\
\gamma(t,t+1) &= \mathbb{E}(W_{t-1} + 2W_t + W_{t+1})(W_t + 2W_{t+1} + W_{t+2}) \\
&= 2\mathbb{E}W_t^2 + 2\mathbb{E}W_{t+1}^2 = 4\sigma_w^2 \\
\gamma(t,t+2) &= \mathbb{E}(W_{t-1} + 2W_t + W_{t+1})(W_{t+1} + 2W_{t+2} + W_{t+3}) \\
&= \mathbb{E}W_{t+1}^2 = \sigma_w^2
\end{aligned}$$

and $\gamma(t,t+h) = 0$ for $h \geq 3$. By symmetry, $\gamma(t,t-h) = \gamma(t,t+h)$.

For the autocorrelation function, we saw above that $\gamma(t,t) = 6\sigma_w^2$ for all t. Therefore,

$$\rho(h) = \frac{\gamma(t,t+h)}{\gamma(t,t)}$$

and so $\rho(0) = 1$, $\rho(1) = 2/3$, $\rho(2) = 1/6$ and $\rho(h) = 0$ for $h \geq 3$.

The plots of the autocorrelation and autocovariance are shown in Figure 1

4. (a) If we differentiate with respect to A, we obtain

$$\begin{aligned}
\frac{d}{dA} MSE(A) &= \frac{d}{dA}(\mathbb{E}X_{t+\ell}^2 + A^2 \mathbb{E}X_t^2 - 2A\mathbb{E}X_t X_{t+\ell}) \\
&= 2A\mathbb{E}X_t^2 - 2\mathbb{E}X_t X_{t+\ell} \\
&= 2A\gamma(0) - 2\gamma(\ell).
\end{aligned}$$

Setting this to zero, we see that the minimum is obtained at $A = \gamma(\ell)/\gamma(0) = \rho(\ell)$.

(b) Plugging $A = \rho(\ell)$ back into the expression for MSE, we have

$$MSE(A) = \gamma(0) + \rho^2(\ell)\gamma(0) - 2\rho(\ell)\gamma(\ell) = \gamma(0)(1 - \rho^2(\ell))$$

since $\gamma(\ell) = \rho(\ell)\gamma(0)$.

5. The plots for this problem are shown in Figure 2

(a) X_t oscillates regularly, with period about 4. This is to be expected because X_t is strongly negatively correlated with X_{t-2}. In V_t, the oscillations are smoothed out.

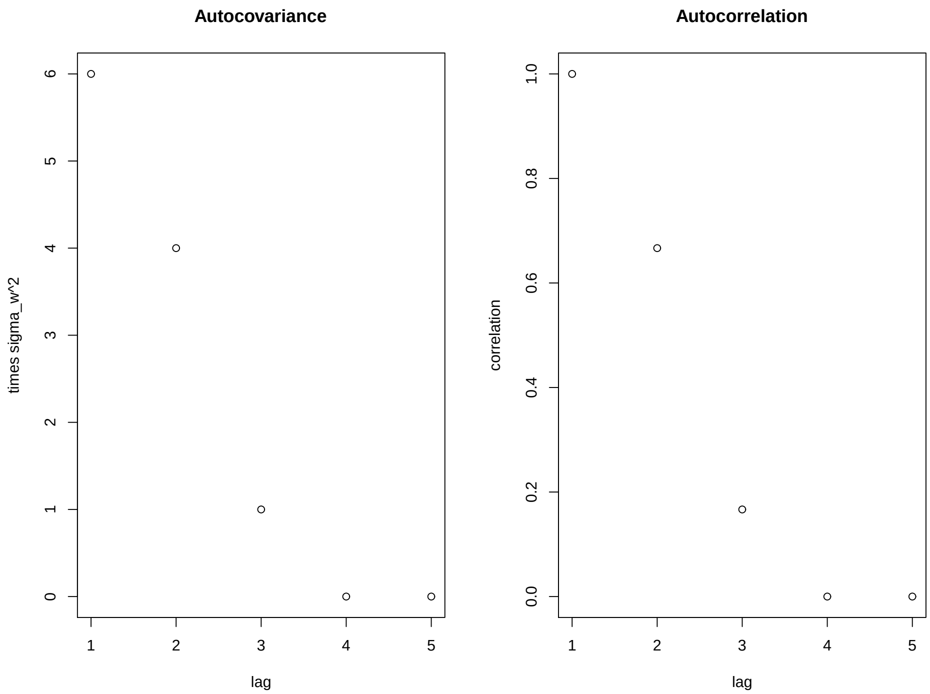

Figure 1: Autocorrelation and autocovariance plots for Problem 3.

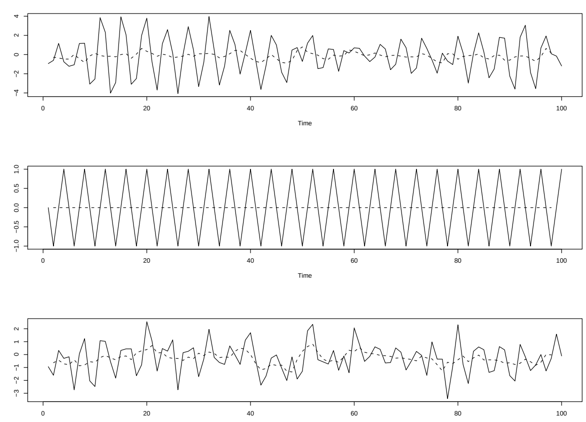

Figure 2: Plots for Problem 5.

(b) X_t oscillates with period 4. Since there is no noise, V_t *completely* smooths out the oscillations, resulting in a flat line.

(c) X_t oscillates more-or-less with period 4, but there is quite a bit of noise. V_t smooths the oscillations.

(d) The same pattern is visible in (a)–(c). In each case, X_t had regular oscillations with period 4, and V_t smoothed out the oscillations, more or less. This was particularly noticeable in part (b) since there was no noise. Part (a) is a little different from the other two because it is not stationary, but this isn't particularly visible from the plots. What is visible, however, is that X_t is strongly correlated with X_{t+4} in part (a), while it isn't in part (c). This can be seen from the fact that the peaks in part (a) vary relatively smoothly.

The R code that generated the data for this problem is as follows:

```
w <- rnorm(100)
xa <- filter(w, filter=c(0, -0.9), method="recursive")
va <- filter(xa, filter=c(1/4, 1/4, 1/4, 1/4), method="convolution")
xb <- cos(2*pi*(1:100)/4)
vb <- filter(xb, filter=c(1/4, 1/4, 1/4, 1/4), method="convolution")
xc <- cos(2*pi*(1:100)/4) + w
vc <- filter(xc, filter=c(1/4, 1/4, 1/4, 1/4), method="convolution")

par(mfcol=c(3,1))
postscript(file="stat_153_solutions1_5.eps")
plot(cbind(xa, va), plot.type="single", lty=1:2)
plot(cbind(xb, vb), plot.type="single", lty=1:2)
plot(cbind(xc, vc), plot.type="single", lty=1:2)
dev.off()
```

6. The plot of the sample autocorrelation function is in Figure 3. The first 7 coefficients are approximately $(1.00, 0.62, 0.13, 0.05, 0.00, -0.14, -0.20)$ and the `R` code that generated the data is as follows:

```
w <- rnorm(102)
x <- filter(w, filter=c(1, 2, 1), method="convolution")[2:101]
postscript(file="stat_153_solutions1_6.eps")
a <- acf(x, type="correlation")
dev.off()

print(a$acf)
```

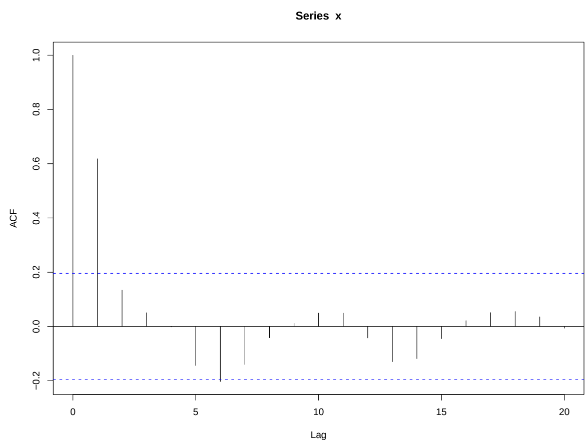

Figure 3: Sample autocorrelation function for Problem 6.