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Percolating paths through random points

• Focus on one particular set of problems (which look easy)

• Digression to different views of “big picture”.

1
Big Picture

Random variable $X_n$ associated with some “size $n$” random structure. Seek to study $EX_n$. Suppose

(i) can’t do useful explicit calculations within size-$n$ model

(ii) know order-of-magnitude, say order $n$.

Guess there is some limit constant $c$

$$n^{-1}EX_n \to c. \quad (1)$$

Two well-known techniques one can use to try to prove (1):

- subadditivity
- weak convergence
**Topic of this talk.** Take a Poisson point process (PPP) of rate 1 in $\mathbb{R}^d$ for $d \geq 2$. There should be some number whose intuitive interpretation is

“smallest possible average edge length in a path through an infinite subset of points of the PPP”.

Analog to critical value in continuum percolation, where a rigorous definition is easy.
We'll discuss 4 possible formalizations.

1. **Short paths from the origin.** For each $m \geq 1$ define a r.v.

   \[ T_m = \text{length of shortest path } 0, \xi_1, \xi_2, \ldots, \xi_m \text{ through } m \text{ distinct points of the PPP.} \]

   Guess: $T_m/m \to \text{constant}$. 

   But is this easy to prove?
2. **Paths across a diagonal.** For $s > 0$ consider the cube $[0, s]^d$ with 0 and $s$ as diagonally opposite vertices. For a path $\pi: 0, \xi_1, \xi_2, \ldots, \xi_m, s$ through distinct points of the PPP in $[0, s]^d$, write

$$m(\pi) = \text{number of points}$$

$$\ell(\pi) = \text{length of path}$$

and then define a r.v.

$$W_s = \min_{\pi} \frac{\ell(\pi)}{m(\pi)}$$

(minimum of average edge-length in a path).

Guess: $W_s \rightarrow \text{constant as } s \rightarrow \infty$.

This definition designed for help with subadditivity.
3. Cycles through a given proportion of points.

Poissonized version of a result going back to Beardwood-Halton-Hammersley (1959) on “the Euclidean TSP”:

Write $N(s)$ for number of points of the PPP in the cube $[0, s]^d$. Define $L_s(1) :=$

\[
\frac{\text{length of shortest cycle through all } N(s) \text{ points}}{N(s)}
\]

(minimum of average edge-length in a tour).

BHH proved (subadditivity argument) $L_s(1) \to c(1)$.

We consider a variation: Define $L_s(\delta) :=$

\[
\frac{\text{length of shortest cycle through some } \lceil \delta N(s) \rceil \text{ points}}{\lceil \delta N(s) \rceil}
\]

(minimum of average edge-length in a sparse cycle).

Guess: $L_s(\delta) \to c(\delta)$ as $s \to \infty$.

Is this easy to prove by subadditivity?

Guess:
the function $\delta \to c(\delta)$ is increasing;
the limit $c(0^+)$ is the limit constant in Formalizations 1 and 2.
Seek definition directly on $\mathbb{R}^d$, as with continuum percolation.

4. **Invariant paths on** $\mathbb{R}^d$. Consider pair $(\mathcal{X}, \mathcal{E})$ where $\mathcal{X}$ is a locally finite point set in $\mathbb{R}^d$ and $\mathcal{E}$ is set of edges $(x_i, x_j)$ with $x \in \mathcal{X}$, these edges forming a collection of doubly-infinite paths. Formalize space $S$ of such pairs — marked point process. Consider a translation-invariant probability measure $\mu$ on $S$ under which the points form a rate-1 PPP. There there exist constants $\delta(\mu), \ell(\mu)$ such that, writing $\mathcal{V}$ for end-vertices of $\mathcal{E}$,

$$E \left| \mathcal{V} \cap [0, s]^d \right| = \delta(\mu) \ s^d$$

$$E \left( \text{length of } \mathcal{E} \cap [0, s]^d \right) = \delta(\mu) \ell(\mu) s^d.$$

Via Palm theory, interpret $\delta(\mu) =$ proportion of the Poisson points which are in some path $\ell(\mu) =$ average edge-length within paths.

Define $\bar{c}(\delta) := \inf \{ \ell(\mu) : \delta(\mu) = \delta \}$

Guess: $\bar{c}(\delta) = c(\delta)$ (from formalization 3).

Easy to prove via weak convergence?
Which of these guesses are in fact easy to prove?

Recall how subadditivity is used in Beardwood-Halton-Hammersley. Same ideas work to prove

\[ L_s(\delta) \to c(\delta) \text{ as } s \to \infty. \]

Moreover there are two cheap tricks:

(i) use proportion \( \delta \) points in some subsquares, 0 in others;
(ii) use proportion \( \delta_1 \) points in some subsquares, proportion \( \delta_2 \) in others

which show

(i) \( \delta \to c(\delta) \) is weakly increasing;
(ii) \( \delta c(\delta) \) is convex.

This implies: either

(a) \( c(\delta) \) is strictly increasing on \( 0 < \delta < 1 \);
or (b) \( c(\delta) \) is constant on some \( 0 < \delta < \delta_0 \).
How to relate “paths across a diagonal” to this?

If we know a limit constant exists for $W_s$, easy to show limit $= c(0+)$. 

One can give general result on “optimal cost/reward ratios” in subadditive settings. The trick is: for constant $\gamma$ the criterion

$$E \min \{\ell(\pi) - \gamma m(\pi) : \pi \text{ path 0 to s} \} \geq 0 \ \forall s$$

determines a critical value $\gamma_0$ which is the limit

$$W_s := \min \{\ell(\pi)/m(\pi) : \pi \text{ path 0 to s} \} \to \gamma_0.$$ 

One can invent many other problems which can be solved this way .......
Short paths from the origin. $T_m =$ length of shortest path through $m$ distinct points of the PPP. Natural approach:

Let’s suppose $T_m/m \to c^*$ where (easy) $c^* \leq c(0+)$.  

Then the **Conjecture** $c^* = c(0+)$ is equivalent to:  

there exist $m$-step paths from the origin, with length $\sim c^*m$, which stay inside ball of radius $o(m)$ (**sublinear growth**).  

Maybe proof requires more sophisticated “percolation” techniques.
Invariant paths on $\mathbb{R}^d$.

easy to justify via \textit{local weak convergence}, which looks at a window around a randomly-chosen origin in the cube $[0, s]^d$.

Letting $s \to \infty$ and considering a subsequential weak limit gives a translation-invariant distribution on points-and-paths.
Summary of “percolating paths through random points”

Easy to prove equivalence of
(2) Paths across a diagonal
(3) Cycles through a given proportion of points
(4) Invariant paths on $\mathbb{R}^d$
and that $c(0+) > 0$ (comparison with branching RW).

Open Problems

• (1) Short paths from the origin?

• $c(\delta)$ strictly increasing?

• $c(\delta) - c(0+) \propto \delta^\alpha$ for some $\alpha$, maybe $\alpha = 1/3$?

• Monte Carlo study of $c(\delta)$?

• $\text{var}(T_m) \asymp m^{2/3}$, or just $o(m)$?
The invariant measure on (collections of) infinite paths fits theme **Stochastic analysis and non-classical random processes.** Can we do calculations with this type of random object?

∃ lots of scattered work on discrete infinite random graphical structures of different kinds . . . . . . . In particular there is a “mean-field” model where one can do explicit calculations.

In our Euclidean setting, no hope for explicit calculation on \( c(\delta) \). But maybe

(i) study strict monotonicity of \( c(\delta) \)

(ii) let \( \delta \to 0 \), do spatial rescaling; guess limit is some continuum self-avoiding path – related to SLE ????