Aldous notes: February 2002
xxx "Up the River" game story
xxx need good name for the process!

First consider $K \geq 1$ particles performing independent Brownian motion on $[0, \infty)$, with state 0 absorbing, and each particle started at state 1 at time 0. Call this the *uncontrolled* process, and write it as $(\widetilde{X}_i(t), 0 \leq t < \infty, 1 \leq i \leq K)$. We study a *controlled process* in which we have at our disposal a unit quantity of positive drift, which we can distribute amongst the non-absorbed particles at time t according to any *control policy* we choose. Thus a controlled process is specified by $(X_i(t), A_i(t), 0 \leq t < \infty, 1 \leq i \leq K)$ where the drift terms A_i must satisfy

$$\begin{aligned} A_i(t) &\geq 0 \\ \sum_i A_i(t) &\leq 1 \\ A_i(t) = 0 \text{ and } X_i(t) = 0 &\qquad \text{if } \inf_{s \leq t} X_i(s) = 0. \end{aligned}$$

In a controlled process, consider the number of particles which are never absorbed

$$N := \#\{i : X_i(t) > 0 \ \forall t\}.$$

(For the uncontrolled process, the corresponding number is of course zero.) Suppose we seek to maximize EN. Then we can ask

- Which control policy maximizes EN?
- What is the resulting maximal value, say $\psi(K)$, of EN?

As the next lemma shows, we can obtain the correct order of magnitude of $\psi(K)$ as $K \to \infty$ without needing to identify and analyze the optimal control.

Lemma 1 *(a)* $\psi(K) \geq (2e^{-1/2} - o(1))K^{1/2}$ *as* $K \to \infty$.
(b) $\psi(K) \leq 5\pi^{-1/2}K^{1/2}$.

Proof. (a) Take $1 \leq m \leq K$ and consider the following control policy. For each particle, assign no drift until it reaches position K/m and (if it does reach that position and is one of the first m particles to do so) thereafter assign drift $1/m$. Clearly this is a permitted control policy. From the standard formulas

$$P_1(\text{ Brownian motion, drift 0, hits } K/m \text{ before hitting } 0) = m/K$$

$$P_x(\text{Brownian motion, drift } \mu > 0, \text{ hits } 0) = \exp(-2\mu x)$$

we calculate

$$EN = \exp(-2K/m^2) \cdot E\min(m, \text{Bin}(K, m/K))$$

where $\mathrm{Bin}(\cdot,\cdot)$ denotes a Binomial random variable. Now let $K \to \infty$ with $m \sim \theta K^{1/2}$, giving

$$EN \sim \theta \exp(-2/\theta^2)\ K^{1/2}.$$

The function $\theta \to \theta \exp(-2/\theta^2)$ takes maximum value $2e^{-1/2}$ at $\theta = 2$, establishing (a).

For (b), consider

$$f_t(x) := P_x\left(\inf_{0 \le s \le t} B_s > 0\right),\ x \ge 0$$

where (B_s) is Brownian motion. Fix $\tau > 0$. If Z_t is Brownian motion with drift μ then

$$df_{\tau-t}(Z_t) = dM_t + \mu g_{\tau-t}(Z_t)$$

where M is some martingale and $g_t(x) := \frac{d}{dx} f_t(x)$. Thus for the controlled process $(X_i(t), A_i(t))$, if we define

$$Y_t := \sum_i f_{\tau-t}(X_i(t)),\ 0 \le t \le \tau$$

then

$$dY_t = dM_t + \sum_i A_i(t) g_{\tau-t}(X_i(t))$$

where M is some martingale. But it is standard that g_t is the density function of the absolute value of a $\mathrm{Normal}(0,t)$ r.v., and this density is maximized at $x = 0$ with maximum value $\sqrt{2/(\pi t)}$. Integrating over $0 \le t \le \tau$ gives

$$Y_\tau - Y_0 \le M_\tau - M_0 + \int_0^\tau \sqrt{2/(\pi(\tau - t))}\ dt$$

and so

$$EY_\tau \le EY_0 + \sqrt{8/\pi}\ \tau^{1/2}.$$

Now

$$Y_0 = K f_\tau(1) \le K\sqrt{2/(\pi\tau)}.$$

Define

$$N(\tau) := \#\{i : X_i(\tau) > 0\}.$$

So $T_\tau = N(\tau)$, and so we have shown

$$EN \le EN(\tau) \le K\sqrt{\frac{2}{\pi\tau}} + \sqrt{\frac{8\tau}{\pi}}. \qquad (1)$$

The right side is minimized at $\tau = 2K$ and gives bound (b).

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0.1 A conjectured fluid limit

From now on we analyze the particular policy

assign drift 1 to the lowest particle.

Write N_K for the number of particles which survive forever. In view of Lemma 1 we expect the limit

$$\lim_{K\to\infty} K^{-1/2} E N_K$$

to exist; we will eventually formulate a conjecture for its value.

First reconsider the uncontrolled process. From the exact formula for $P_1(B_t \in \cdot, \inf_{0\leq s\leq t} B_s > 0)$ is it straightforward to deduce the following "fluid limit" result. As $K \to \infty$, for each $y \geq 0, t > 0$,

$$K^{-1/2}\#\{i : \widetilde{X}_i(tK) \leq yK^{1/2}\} \to_p \int_0^y \tilde{f}(t,x)\ dx \tag{2}$$

where

$$\tilde{f}(t,x) := \sqrt{2/\pi} x t^{-3/2} \exp(-x^2/(2t)). \tag{3}$$

Moreover $\tilde{f}$ is a solution of the heat equation

$$\frac{d}{dt} f = \frac{1}{2} \frac{d^2}{dx^2} f \tag{4}$$

with the (absorbing) boundary condition

$$\tilde{f}(t,0) = 0.$$

And the total mass $\tilde{F}(t) := \int_0^\infty \tilde{f}(t,x) dx$ satisfies

$$\tilde{F}(t) = \sqrt{2/\pi}\ t^{-1/2}.$$

We now start to formulate a conjecture that the optimally controlled process behaves in a qualitatively similar way, in that there is a fluid limit

$$K^{-1/2}\#\{i : X_i(tK) \leq yK^{1/2}\} \to_p \int_0^y f(t,x)\ dx \tag{5}$$

for a certain $f(x,t)$. Intuitively, f must also satisfy the heat equation and should have similar small-t behavior to the uncontrolled process, say

$$f(t,x) \sim \tilde{f}(t,x) \text{ as } t \to 0 \text{ with } x \sim at^{1/2},\ a > 0. \tag{6}$$

The effect of the control is to change the (left) boundary behavior, in (we conjecture) the following way. Let $b(t)$ be a boundary of the form

$$b(t) = 0,\ 0 \leq t \leq t_0; \quad b(t) > 0,\ t > t_0 \tag{7}$$

for some t_0. Then the boundary condition is

$$f(t, b(t)) = 2, \quad 0 \leq t < \infty. \tag{8}$$

Why the constant should be 2 is explained below Conjecture 3. Here is a more careful statement.

Conjecture 1 *(a) There is a unique choice of $b(t)$ of form (7) for some t_0, and of $f(t,x)$ defined on the region $D := \{(t,x) : t \geq 0, b(t) \leq x\}$, such that f satisfies the heat equation (4) on the interior of D, and satisfies the initial condition (6) and the boundary condition (8) and the further boundary condition $\frac{d}{dx} f(t,x)|_{x=b(t)} = 0$ for $t > t_0$.*
(b) This f is the fluid limit of the controlled process, in the sense (5).

Before explaining why we expect this particular boundary behavior, let us explain how (remarkably) the qualitative behavior asserted in Conjecture 1 leads to a quantitative conjecture. Reconsider the argument leading to the inequality (1) for $N(\tau)$. For large K, the only reason this is an inequality (rather than an essential equality) is the fact $g_{\tau-t}(X_1(t)) < g_{\tau-t}(0)$, that is the fact that $X_1(t) > 0$. But the qualitative behavior in Conjecture 1 implies that for $t \leq t_0 K$ the position $X_1(t)$ of the left-most particle will be close to 0 (relative to the $K^{1/2}$ spatial rescaling), and the upshot is that (1) becomes an approximate equality

$$EN(\tau) \approx K\sqrt{\frac{2}{\pi\tau}} + \sqrt{\frac{8\tau}{\pi}}; \quad \tau \leq t_0 K. \tag{9}$$

Moreover the qualitative behavior of the boundary $b(t)$ in Conjecture 1 implies that only a negligible number of particles will be absorbed after time $t_0 K$;

$$EN \approx EN(t_0 K). \tag{10}$$

Now recall that the right side of (9) is mimimized at $\tau = 2K$. Since $EN(\tau)$ is *a priori* decreasing, (9) implies we cannot have $t_0 > 2$. And since $EN \leq EN(2K)$, (10) implies we cannot have $t_0 < 2$. In other words, we must have $t_0 = 2$, and the upper bound derived from (1) must be asymptotically correct.

Conjecture 2 *(i)* $\lim_K K^{-1/2} EN_K = 5\pi^{-1/2}$.
(ii) In Conjecture 1 we have $t_0 = 2$.

0.2 Stationary distributions for the controlled particle process

Now switch viewpoints and consider the optimally-controlled process as an interacting particle process with *infinitely* many particles. That is, particles perform independent Brownian motion on $[0,\infty)$, with state 0 absorbing, and with the leftmost non-absorbed particle being given drift rate 1. A natural initial distribution is $\mathcal{P}_\mu$, the Poisson (rate μ) point process on $[0,\infty)$. Let $0 < \xi_1(t) < \xi_2(t) < \ldots$ denote the positions of non-absorbed particles at time t.

Conjecture 3 *(a) Under $\mathcal{P}_\mu$ initial distribution, for any $0 < \mu < \infty$, we have convergence in distribution*

$$(\xi_i(t) - \xi_1(t), i \geq 1) \to_d (\eta_i, i \geq 1)$$

where the limit distribution of $0 = \eta_1 < \eta_2 < \eta_3 < \ldots$ does not depend on μ and has "density 2" in the sense

$$\eta_m/m \to_p 1/2 \text{ as } m \to \infty. \tag{11}$$

(b) If $\mu < 2$ then $\xi_1(t) \sim c_\mu t^{1/2}$, where $c_\mu := (1 - \frac{\mu}{2})\sqrt{2/\mu} > 0$.
(c) If $\mu > 2$ then $\xi_1(t) = o(t^{1/2})$.

Let us first explain how Conjecture 3 suggests the boundary behaviour asserted in Conjecture 1. Consider the K-particle process at time tK; presumably this has some density μ of particles over the spatial interval $[0, L]$ where $1 \ll L \ll K^{1/2}$. Continue until time $(t+\varepsilon)K$: this time increment $\varepsilon K^{1/2}$ is large in absolute terms, so the "relaxation to equilibrium" in Conjecture 3 says that this density must become either 2 (if $\mu > 2$) or 0 (if $\mu < 2$). Thus in the fluid limit we expect f to have $f(t, 0) = 2$ as long as the derivative $\frac{d}{dx} f(t, x)|_{x=0} > 0$. Time t_0 should then be the first time at which $\frac{d}{dx} f(t, x)|_{x=0} = 0$. Thereafter the boundary $b(t)$ behaves in such a way as to make $f(t, b(t)) = 2$ and $\frac{d}{dx} f(t, x)|_{x=b(t)} = 0$, the latter being the "reflecting barrier" boundary condition (for reasons described later).

0.3 Heuristics for Conjecture 3

The central idea is "2 is the critical density", in a certain sense (so assertion (a) could be viewed as "self-organized criticality". But I digress!).

Suppose we initially have $\mathcal{P}_\mu$ distribution, and suppose we let the particles perform independent Brownian motions without boundary or control. Fix t. The time-t distribution is a certain non-homogeneous Poisson process. If we want to move particles to restore the initial distribution, how much "work" (sum of distances moved, over all particles) do we need? This is easy by a trick; if we had a *reflecting* boundary at 0 the process would be stationary. So the reflection principle says we can just move any particle ending at $-y$ to $+y$. Thus the mean "work" required is

$$\int_0^\infty \mu dx \ 2 \ E(B_t - x)^+ = \mu t/2.$$

Because we have drift 1 available, the amount of work available in time t equals t. Thus $\mu = 2$ is "critical" in that, to first order, the drift offsets the diffusive tendancy. We expect the stationary distribution (η_i) to be something like the $\mathcal{P}_2$ distribution.

For assertion (c), if $\mu < 2$ then we have "extra push" from the drift which will push the ensemble of particles away from 0. We expect that at time t, there is some $c_+(t)$ such that the initial $\mathcal{P}_\mu$ distribution on $[0, c_+(t)]$ has become approximately a $\mathcal{P}_2$ distribution on some $[c_-(t), c_+(t)]$. To conserve particles we must have

$$\mu c_+(t) = 2(c_+(t) - c_-(t)) \Rightarrow c_-(t) = (1 - \mu/2)c_+(t).$$

The excess work must be enough to move these particles the required distance; this gives

$$t - \mu t/2 = \mu c_+(t) \times c_-(t)/2.$$

Solving for $c_-(t)$ gives the formula in (c).

As for (b), the actual behavior of $\xi_1(t)$ in the case $\mu > 2$ seems rather subtle, since for trivial reasons (asymptotic rate of absorbed particles) we expect $\xi_1(t) \to_p \infty$ while the sample path of $\xi_1(t)$ must hit 0 infinitely often.

Another part of this big picture is that we expect the limit stationary process $(\eta_i(t))$ to be qualitatively like the $\mathcal{P}_2$ stationary Brownian process with *reflecting* barrier. (Hence the "reflecting barrier" condition in the fluid limit).