Patterns in random walks and Brownian motion

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Question

Given some distribution of a process X with continuous paths, is there a random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as $(X_u, 0 \leq u \leq 1)$?.

- Examples : Brownian/pseudo bridge, Brownian meander, normalized excursion, Bessel(3), Vervaat bridges. . . etc.
- The question here has some affinity to the well-known ***Skorokhod embedding problem***.
- The question is related to ***splitting theorems*** of post-T Markov processes.

Scenario #2

Question

Given a Borel measurable subset $S \subset \mathcal{C}[0,1]$, can we find a random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S$ with probability one ?

Examples :

$$\mathcal{E} := \{w \in \mathcal{C}[0,1]; w(t) > w(1) = 0 \text{ for } 0 < t < 1\};$$

$$\mathcal{M} := \{w \in \mathcal{C}[0,1]; w(t) > 0 \text{ for } 0 < t \leq 1\};$$

$$\mathcal{BR}^{\lambda} := \{w \in \mathcal{C}[0,1]; w(1) = \lambda\};$$

$$\mathcal{FP}^{\lambda} := \{w \in \mathcal{C}[0,1]; w(t) > w(1) = \lambda \text{ for } 0 \leq t < 1\};$$

$$\mathcal{VB}^{\lambda} := \{w \in \mathcal{FP}^{\lambda}; \zeta := \inf\{t > 0; w(t) < 0\} > 0\} \ldots \text{etc.}$$

Scenario #3

Question

Given for each n a collection $\mathcal{A}^n$ of patterns of length n, what is the order of the expected waiting time $\mathbb{E}T(\mathcal{A}^n)$ until one of the elements of $\mathcal{A}^n$ is observed in a random walk ?

Examples :

$$\mathcal{E}^{2n} := \{w \in SW(2n); w(i) > 0 \text{ for } 1 \leq i \leq 2n-1 \text{ and } w(2n) = 0\};$$

$$\mathcal{M}^{2n+1} := \{w \in SW(2n+1); w(i) > 0 \text{ for } 1 \leq i \leq 2n+1\};$$

$$\mathcal{BR}^{\lambda,n} := \{w \in SW(n); w(n) = \lambda_n\} \quad \textit{where } \lambda_n \sim \lambda\sqrt{n};$$

$$\mathcal{FP}^{\lambda,n} := \{w \in SW(n); w(i) > w(n) = \lambda_n \text{ for } 0 \leq i \leq n-1\} \ldots \text{etc.}$$

Response to Scenario #3

Theorem

1. *There exists* $C_{\mathcal{E}} > 0$ *such that*
$$\mathbb{E}T(\mathcal{E}^{2n}) \sim C_{\mathcal{E}} n^{\frac{3}{2}};$$
2. *There exists* $C_{\mathcal{M}} > 0$ *such that*
$$\mathbb{E}T(\mathcal{M}^{2n+1}) \sim C_{\mathcal{M}} n;$$
3. *There exists* $C^{\lambda}_{\mathcal{BR}} > 0$ *such that*
$$\mathbb{E}T(\mathcal{BR}^{\lambda,n}) \sim C^{\lambda}_{\mathcal{BR}} n;$$
4. *There exists* $c^{\lambda}_{\mathcal{FP}}$ *and* $C^{\lambda}_{\mathcal{FP}} > 0$ *such that*
$$c^{\lambda}_{\mathcal{FP}} n \leq \mathbb{E}T(\mathcal{FP}^{\lambda,n}) \leq C^{\lambda}_{\mathcal{FP}} n^{\frac{5}{4}}.$$

Response to Scenario #2

Theorem

1. *a.s.* $\nexists$ *random time* T *such that* $(B_{T+u} - B_T; 0 \leq u \leq 1) \in \mathcal{E} := \{w \in \mathcal{C}[0,1]; w(t) > w(1) = 0 \text{ for } 0 < t < 1\}$;
2. *a.s.* $\nexists$ *random time* T *such that* $(B_{T+u} - B_T; 0 \leq u \leq 1) \in \mathcal{RBR} := \{w \in \mathcal{C}[0,1]; w(t) \geq w(1) = 0 \text{ for } 0 \leq t \leq 1\}$;
3. *For each* $\lambda < 0$, *a.s.* $\nexists$ T *s.t.* $(B_{T+u} - B_T; 0 \leq u \leq 1) \in \mathcal{VB}^{\lambda} := \{w \in \mathcal{FP}^{\lambda}; \zeta := \inf\{t > 0; w(t) < 0\} > 0\}$, *where* $\mathcal{FP}^{\lambda} := \{w \in \mathcal{C}[0,1]; w(t) > w(1) = \lambda \text{ for } 0 \leq t < 1\}$.

Consequence : no ***normalized excursion***, ***reflected bridges***, ***Vervaat bridges*** in a Brownian path !
Idea : ***Williams' path decompositions***, or ***fragmentation argument***.

Response to Scenario #1

Theorem

For each of the following three processes $X := (X_u, \leq u \leq 1)$ *there is some random time* T *such that* $(B_{T+u} - B_T; 0 \leq u \leq 1)$ *has the same distribution as* X *:*

1. *the meander* $X = (m_u; 0 \leq u \leq 1)$ *;*
2. *the co-meander* $X = (\widetilde{m}_u; 0 \leq u \leq 1)$ *;*
3. *the Bessel*(3) *process* $X = (R_u; 0 \leq u \leq 1)$.

Idea : Brownian meander by ***Itô's excursion theory*** and the other two by ***acceptance-rejection method***.

Our favorite open problem

Open problem

Can we find a random time T such that $(B_{T+u} - B_T; 0 \le u \le 1)$ has the same distribution as Brownian bridge $(b^0_u; 0 \le u \le 1)$?

The bridge pattern $\mathcal{BR}^0$ is achieved by the bridge-like process $(B_{T+u} - B_T; 0 \le u \le 1)$, where $T := \inf\{t > 0; B_t - B_{t+1} = 0\}$.

- The bridge-like process can be inferred from the work of Slepian and Shepp $\Longrightarrow \mathbb{P}(T > t)$ is computed.
- From simulation, the above bridge-like process is not Brownian bridge.
- The related Slepian zero set has rich properties, work in progress with Jim Pitman.

Thank you for your attention,
AND
Happy birthday, Jim !