Shifting processes with cyclically exchangeable increments at random

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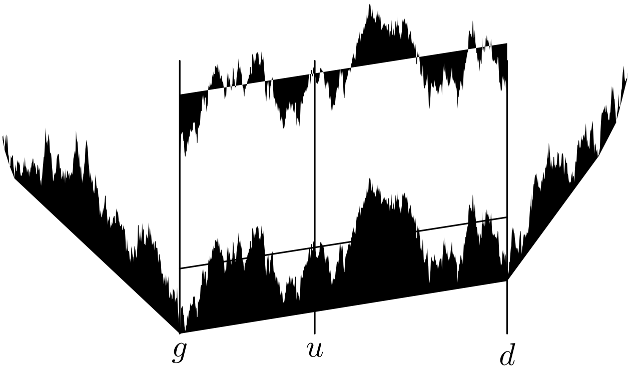
Combinatorial Stochastic Processes
A conference in celebration of Jim Pitman's work
UC San Diego, June 21, 2014

A question of Jim regarding the Vervaat transformation



Question

Is a general framework allowing to interpret the Vervaat transformation as conditioning a process with to remain positive (or above a line)?

- J. Abramson, J. Pitman, N. Ross, and G. Uribe Bravo, *Convex minorants of random walks and Lévy processes*, Electronic Communications in Probability **16** (2011), 423–434

The normalized Brownian excursion and Vervaat's theorem

Let X be a Brownian bridge from 0 to 0 of length 1.

Definition

There exists a unique weakly continuous family of laws $\{\mathbb{P}_y^1 : y \in \mathbb{R}\}$ which is a version of the conditional law of a Brownian motion on $[0, 1]$ given that it ends at y.

The normalized Brownian excursion and Vervaat's theorem

Let X be a Brownian bridge from 0 to 0 of length 1.

Theorem [DIM77] and [Ver79]

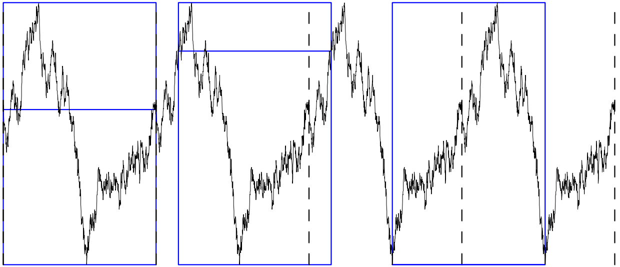
The law of X conditioned to remain above $-\varepsilon$ converges weakly as $\varepsilon \to 0$ toward law of the normalized Brownian excursion. Furthermore, the weak limit can also be constructed as follows: if ρ is the unique instant at which X attains its minimum, then the weak limit has the same law as

$$\theta_\rho(X)_t = X_{\{\rho+t\}} - X_\rho.$$

(Interchanges the paths $(X_s, s \leq \rho)$ and $(X_s, s \geq \rho)$. Starts and ends at zero.)

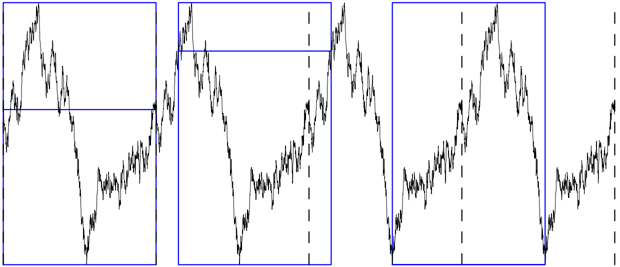
- Richard T. Durrett, Donald L. Iglehart, and Douglas R. Miller, *Weak convergence to Brownian meander and Brownian excursion*, Ann. Probability **5** (1977), no. 1, 117–129. MR 0436353
- Wim Vervaat, *A relation between Brownian bridge and Brownian excursion*, Ann. Probab. **7** (1979), no. 1, 143–149. MR 515820

Idea of proof



$$\theta_t X(s) = X_{s+t} - X_t \qquad \underline{X} = \min_{s\leq 1} X_s \qquad X_\rho = \underline{X} \qquad \underline{X} \circ \theta_t = \underline{X} - X_t.$$

Idea of proof



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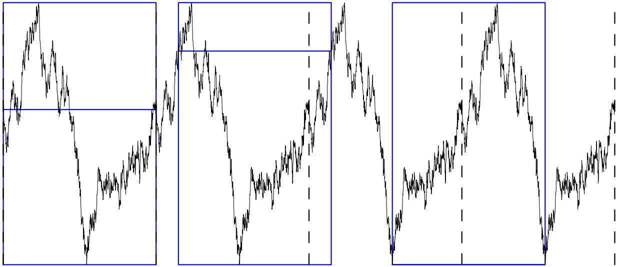
Choose η uniformly on $\{\underline{X} \circ \theta_t \in (-\varepsilon, 0)\}$.

Idea of proof



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Let F^ε be the random distribution

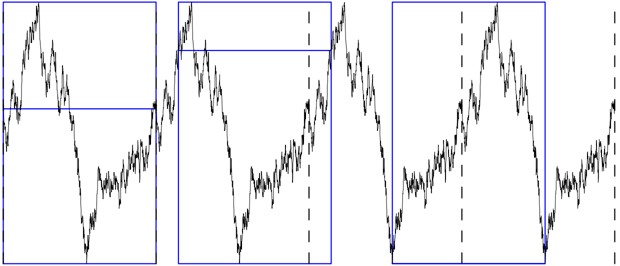
Choose η uniformly on $\{\underline{X} \circ \theta_t \in (-\varepsilon, 0)\}$.

$$F^\varepsilon_t = \frac{\int_0^t \mathbf{1}_{X_s - \underline{X} < \varepsilon}\, ds}{\int_0^1 \mathbf{1}_{X_s - \underline{X} < \varepsilon}\, ds}.$$

$U \perp X$: uniform

$\tau_t = \inf\{s \ge 0 : F^\varepsilon_s > t\} \quad$ and $\quad \eta = \tau_U.$

Idea of proof



$$\theta_t X(s) = X_{s+t} - X_t \qquad \underline{X} = \min_{s\le 1} X_s \qquad X_\rho = \underline{X} \qquad \underline{X}\circ\theta_t = \underline{X} - X_t.$$

Choose η uniformly on $\{\underline{X}\circ\theta_t \in (-\varepsilon, 0)\}$.
Then:

- the law of $\theta_\eta X$ equals the law of X conditioned on $\underline{X} \in (-\varepsilon, 0)$.
- $\eta \to \rho$ as $\varepsilon \to 0$.

Let F^ε be the random distribution

$$F^\varepsilon_t = \frac{\int_0^t \mathbf{1}_{X_s - \underline{X} < \varepsilon}\, ds}{\int_0^1 \mathbf{1}_{X_s - \underline{X} < \varepsilon}\, ds}.$$

$U \perp X$: uniform

$$\tau_t = \inf\{s \ge 0 : F^\varepsilon_s > t\} \quad \text{and} \quad \eta = \tau_U.$$

Conditioning the minimum of a CEI process

Definition: CEI process

A càdlàg process has **cyclically exchangeable increments (CEI)** if:

$$\theta_t X \stackrel{d}{=} X \text{ for every } t \in [0,1].$$

Shift of a CEI process:

$$\theta_t X(s) = X_{\{s+t\}} - X_t + X_{\lfloor s+t \rfloor}.$$

(Interchanges the paths $(X_s, s \leq t)$ and $(X_s, s \geq t)$ with same starting and ending points.)

Conditioning the minimum of a CEI process

Let $I \subset (-\infty, 0]$.
To condition on $\{\underline{X} \in I\}$, choose t uniformly on $\{t : \underline{X} \circ \theta_t \in I\}$ using

$$A_t^I = \int_0^t \mathbf{1}_{\underline{X} \circ \theta_s \in I} \, ds.$$

Theorem

Let $(X, \mathbb{P})$ be any non trivial CEI process.
$X_0 = 0$, $X_1 \geq 0$ and $\mathbb{P}(\underline{X} \in I) > 0$.
Let $U \perp X$ be uniform and define:

$$\nu = \inf\{t : A_t^I = U A_1^I\}. \tag{1}$$

Conditionally on $A_1^I > 0$, $\theta_\nu(X)$ is independent of ν and has the same law as X conditionally on $\underline{X} \in I$. Moreover the time ν is uniformly distributed over $[0, 1]$.

Conditioning the minimum of a CEI process

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Theorem

Let $(X, \mathbb{P})$ be any non trivial CEI process.
$X_0 = 0$, $X_1 \geq 0$ and $\mathbb{P}(\underline{X} \in I) > 0$.
Let $U \perp X$ be uniform and define:

$$\nu = \inf\{t : A_t^I = U A_1^I\}. \tag{1}$$

Conversely, if Y has the law of X conditioned on $\underline{X} \in I$ and U is uniform and independent of Y then $\theta_U(Y)$ has the same law as X conditioned on $A_1^I > 0$.

Conditioning the minimum of a CEI process

Corollary

Let $(X, \mathbb{P})$ be any non trivial CEI process such that $X_0 = 0 = X_1$. Assume that there exists a unique $\rho \in (0,1)$ such that $X_\rho = \underline{X}$ and that $X_{\rho-} = X_\rho$. Then, the law of X conditioned to remain above $-\varepsilon$ converges weakly in the Skorohod J_1 topology as $\varepsilon \to 0$. Furthermore, the weak limit is the law of $\theta_\rho X$.

Loïc Chaumont and Gerónimo Uribe Bravo, *Shifting processes with cyclically exchangeable increments at random*, 2014, arXiv:1405.1335

Conditioning on the minimum of an EI process

Definition

A càdlàg stochastic process has **exchangeable increments (EI)** if

$$X_{k/n} - X_{(k-1)/n}, 1 \leq k \leq n$$

are exchangeable for any $n \geq 1$.

$$X_t = \alpha t + \sigma b_t + \sum_i \beta_i \left[\mathbf{1}_{U_i \leq t} - t\right]$$

1. α, σ and $\beta_i, i \geq 1$ are (possibly dependent) rv and $\sum_i \beta_i^2 < \infty$.
2. b is a Brownian bridge
3. $(U_i, i \geq 1)$ are iid uniform random variables on $(0, 1)$.

(α, β, σ) are its canonical parameters.

Olav Kallenberg, *Canonical representations and convergence criteria for processes with interchangeable increments*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **27** (1973), 23–36. MR 0394842

Conditioning on the minimum of an EI process

Application to EI processes

Let X be an EI process with canonical parameters (α, β, σ). On the set

$$\left\{ \sum_i |\beta_i| = \infty, \text{ and } \sum_i \beta_i^2 \left| \log |\beta_i| \right|^{c} < \infty \text{ for some } c > 1 \right\} \cup \{\sigma \neq 0\},$$

X reaches its minimum continuously at a unique $\rho \in (0, 1)$.

The case of Lévy processes

If X is a Lévy process and neither X nor $-X$ is a subordinator: then X achieves its minimum continuously if and only if 0 is regular for $(0, \infty)$ and $(-\infty, 0)$.

To reach the minimum continuously, we assume

H1 0 is regular for $(-\infty, 0)$ and $(0, \infty)$.

The case of Lévy processes

To reach the minimum continuously, we assume

H1 0 is regular for $(-\infty, 0)$ and $(0, \infty)$.

To build bridges, we use transition densities:

H2 For any $t > 0$, $\int \left|\mathbb{E}\left(e^{iuX_t}\right)\right| \, du < \infty$.

Definition

Under **H**, there is an unique weakly continuous family $\left(\mathbb{P}^1_{0,y}, y \in \mathbb{R}\right)$ which is a version of the law of X on $[0, 1]$ given X_1.
The Lévy bridge from 0 to 0 of length 1 has the law $\mathbb{P}^1_{0,0}$.

The case of Lévy processes

To reach the minimum continuously, we assume

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To build bridges, we use transition densities:

H2 For any $t > 0$, $\int \left|\mathbb{E}\left(e^{iuX_t}\right)\right| \, du < \infty$.

Proposition

Assume **H**.
The law $\mathbb{P}_{0,0}^{1}$ has the EI property.
Under $\mathbb{P}_{0,0}^{1}$, the minimum is achieved at a unique place $\rho \in (0, 1)$ and X is continuous at ρ.

The case of Lévy processes

To reach the minimum continuously, we assume

H1 0 is regular for $(-\infty, 0)$ and $(0, \infty)$.

To build bridges, we use transition densities:

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Proposition

Assume **H**.
The law $\mathbb{P}^1_{0,0}$ has the EI property.
Under $\mathbb{P}^1_{0,0}$, the minimum is achieved at a unique place $\rho \in (0, 1)$ and X is continuous at ρ.

Remark

The finite-dimensional distributions of $\theta_\rho X$ are identified with those of the bridge of length 1 from 0 to 0 of the Lévy process conditioned to stay positive in

Gerónimo Uribe Bravo, *Bridges of Lévy processes conditioned to stay positive*, Bernoulli **20** (2014), no. 1, 190–206. MR 3160578

Remarks on the Brownian case

Corollary (Theorem 7 in [BCP03])

Let $\mathbb{P}$ be the law of a Brownian bridge from 0 to 0 of length 1 and let U be uniform and independent of X. Let $\nu = \inf \{t \geq 0 : X_t > U [x + \underline{X}]\}$. Then $\theta_\nu X$ has the same law as the three-dimensional Bessel bridge from 0 to x of length 1.

Corollary

Let $\mathbb{P}$ be the law of the Brownian bridge from 0 to 0 of length 1, let $(L^y_t, y \in \mathbb{R}, t \in [0, 1])$ be its continuous family of local times and let U be uniform and independent of X. For $y \leq 0$, let

$$\eta_y = \inf \left\{ t \geq 0 : L_t^{\underline{X}-y} > U L_1^{\underline{X}-y} \right\}.$$

Then the laws of $X \circ \theta_{\eta_y}$ provide a weakly continuous disintegration of $\mathbb{P}$ given $\underline{X} = y$.

Jean Bertoin, Loïc Chaumont, and Jim Pitman, *Path transformations of first passage bridges*, Electron. Comm. Probab. **8** (2003), 155–166 (electronic). MR 2042754

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Gerónimo Uribe Bravo, *Bridges of Lévy processes conditioned to stay positive*, Bernoulli **20** (2014), no. 1, 190–206. MR 3160578

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