SPECTRAL DYNAMICS OF RANDOM REGULAR GRAPHS AND THE POISSON FREE FIELD

Soumik Pal

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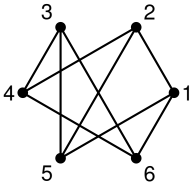
GRAPHS AND ADJACENCY MATRICES

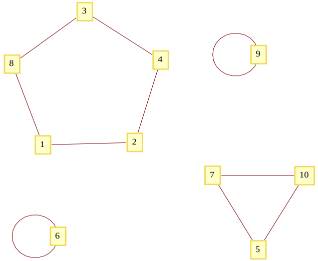
- Undirected graphs on *n* labeled vertices.
- Regular: degree *d*.
- Adjacency matrix = $n \times n$ symmetric matrix.
- *Sparse* - $d \ll n$.



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Models of random regular graphs

- The *permutation* model: $G(n, 2)$.
- π - random permutation on $[n]$.
- 2-regular graph:



THE PERMUTATION MODEL $G(n, 2d)$

$\pi_1, \ldots, \pi_d$ iid uniform permutations. Superimpose.

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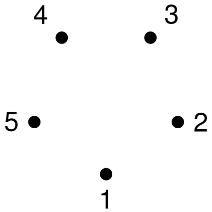

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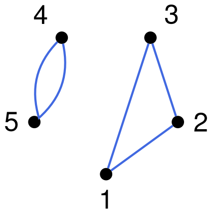

$\pi_1 = (1\ 3\ 2)(4\ 5)$
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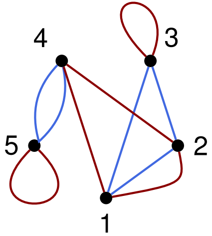

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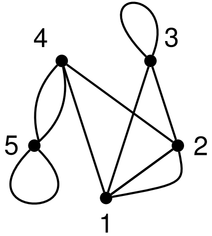

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Multiple edges, loops OK.

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with

Random Matrix Theory

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- upper triangular entries chosen iid $N(0,1)$;

$$\begin{pmatrix} -0.6 & 0.7 & 0.1 & 0.3 \\ & 2.1 & 2.5 & -0.1 \\ & & -2.2 & 1.1 \\ & & & 0.4 \end{pmatrix}$$

A sample of a 4×4 GOE matrix and its 3×3 minor.

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- Minor=principal submatrix, also GOE.

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GOE VS. RANDOM GRAPHS

- Adjacency matrices are not GOE (or, Wigner).
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GOE vs. random graphs

- Adjacency matrices are not GOE (or, Wigner).
- Rows are sparse; no independence.
- However, for large d, approximately GOE.
- Eigenvalue distribution (McKay '81, Dumitriu-P. '10, Tran-Vu-Wang '10)
- Linear eigenvalue statistics (Dumitriu-Johnson-P.-Paquette '11)
- Simulations.
- Not Erdős-Rényi, e.g. connected.

Eigenvalue fluctuations

- W_∞ - GOE array.
- W_n - $n \times n$ minor. E-values $\{\lambda_i^n\}$.
- Linear eigenvalue statistics

$$\mathrm{tr}\, f(W_n) := \sum_{i=1}^{n} f\left(\frac{\lambda_i^n}{2\sqrt{n}}\right).$$

- (Classical Theorem) If f is analytic

$$\lim_{n\to\infty} [\mathrm{tr}\, f(W_n) - \mathbb{E}\, \mathrm{tr}\, f(W_n)] = \mathrm{N}\left(0, \sigma_f^2\right).$$

Dynamics of eigenvalue fluctuations

(A. Borodin '10)

- GOE array $W_\infty(s)$ in time with entries as Brownian motions.
- Choose $(t_i, s_i, f_i,\ i = 1, \ldots, k)$. Polynomial f_i's.

$$\lim_{n\to\infty} \left(\mathrm{tr}\, f_i\left(W_{\lfloor nt_i \rfloor}(s_i)\right) - \mathbb{E}\,\mathrm{tr}\, f_i\left(\cdot\right),\ i \in [k]\right) = \text{Gaussian.}$$

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Fix s. Limiting Height Function is the Gaussian Free Field.
Nontrivial correlation across s.

Main question

What dynamics on random regular graphs
leads to similar eigenvalue fluctuations
in dimension $\times$ time?

Description of the dynamics

Dynamics in dimension

- (Dubins-Pitman) Chinese restaurant process on d permutations.
- ith customers arrive simultaneously. Sits independently.
- Let $T_i = \text{Exp}(i)$, $i \in \mathbb{N}$,

$$n_t = \max\left\{ m : \sum_{i=1}^{m} T_i \le t \right\}.$$

- $G(t, 0) := G(n_t, 2d)$, for $0 \le t \le T$.
- dimension t; time 0.

Dynamics in time

- Fix T large. d permutations on n labels.
- Run random transposition MC simultaneously.
- Any $\binom{n}{2}$ transposition selected at rate $1/n$.
- Successive product on left.
- Superimpose - $G(T, s)$ for $s \geq 0$.
- Delete labels successively:

$$G(T + t, s), \quad t \in [-T, 0], \; s \geq 0.$$

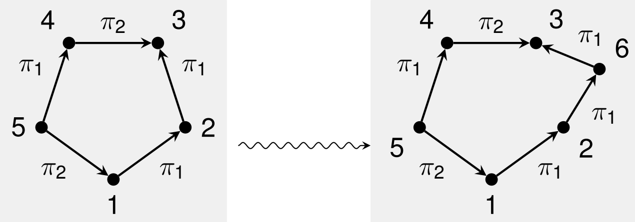
Cycles and eigenvalues

- N_k - # k-cycles in the graph $G(n, 2d)$.
- As $n \to \infty$, $(N_k,\ k \in \mathbb{N})$ - linear eigenvalue statistics.
- In fact
$$2kN_k \approx \mathrm{tr}\left(T_k\left(G(n, 2d)\right)\right).$$
- $(T_k,\ k \in \mathbb{N})$ - Chebyshev polynomials of first kind.

Dynamics of cycles in dimension

Growth of a cycle

(Johnson-P. ’12) Existing cycles grow in size.



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$\pi_1 = (1\ 2\ \mathbf{6}\ 3)(4\ 5)$

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Figure : Vertex 6 is inserted between 2 and 3 in π_1.

Birth of a cycle

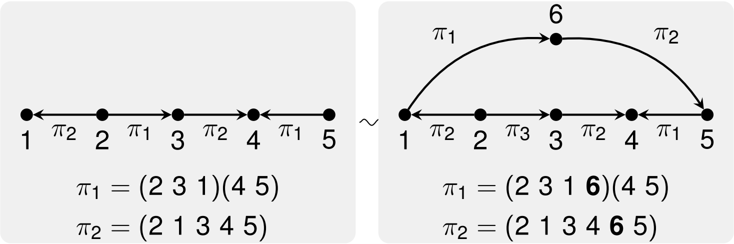

Figure : A cycle forms “spontaneously”.

Cycle counts

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- Non-Markovian process in t, with T fixed.
- $(C_k^{(T)}(t), k \in \mathbb{N},\ t < 0)$ converges as $T \to \infty$.
- Limiting process $(N_k(t),\ k \in \mathbb{N},\ t \leq 0)$ is Markov.
- Running in stationarity.

THE LIMITING PROCESS

(Johnson-P. '12) In the limit:

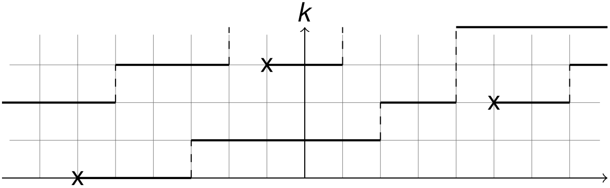
- Existing k-cycles grows to $(k+1)$ at rate k.
- New k-cycles created at rate $\mu(k) \otimes$ Leb.
- Here:

$$\mu(k) = \frac{1}{2}\left[a(d,k) - a(d,k-1)\right], \quad k \in \mathbb{N}, \quad a(d,0) := 0,$$

where

$$a(d,k) = \begin{cases} (2d-1)^k - 1 + 2d, & k \text{ even}, \\ (2d-1)^k + 1, & k \text{ odd}. \end{cases}$$

Poisson field of Yule processes



- Poisson point process χ on $\mathbb{N} \times (-\infty, \infty)$. Intensity $\mu \otimes$ Leb.

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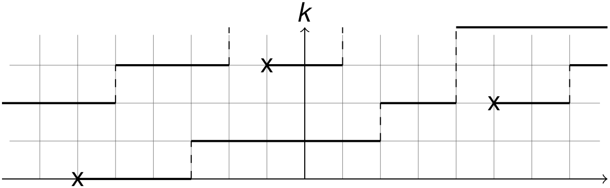

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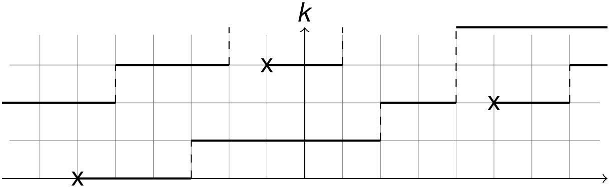

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- For $(k, y) \in \chi$, start indep Yule processes $(X_{k,y}(t),\ t \geq 0)$.
- Define

$$N_k(t) := \sum_{(j,y) \in \chi \cap \{[k] \times (-\infty, t]\}} 1\left\{X_{j,y}(t-y) = k\right\}.$$

Invariant distribution

- $\left(C_k^{(T)}(t), k \in \mathbb{N},\ t \in (-\infty, 0]\right) \longrightarrow (N_k(t), k \in \mathbb{N},\ t \in (-\infty, 0]).$
- Marginal distribution:

$$(N_k(t), k \in \mathbb{N}) \sim \otimes \mathrm{Poi}\left(\frac{a(d,k)}{2k}\right).$$

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- Dumitriu-Johnson-P.-Paquette '11
- Bollobás '80, Wormald '81.

Cycles in time

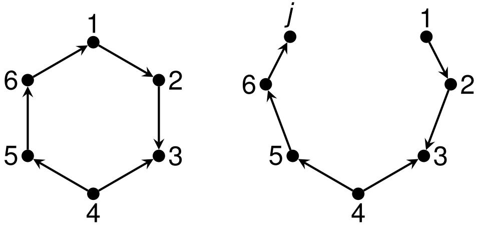

FIGURE : A cycle that vanishes due to transposition $(1, j)$, $j > 6$.

- Random transpositions make short cycles vanish or appear at random.
- Other effects are of negligible probability.

The joint limiting process

(Ganguly-P. '14) Take limit as $T \to \infty$.

- Fix $t < 0$. Consider in $s \geq 0$.
- $(N_k(t, \cdot),\ k \in \mathbb{N})$ - independent birth-and-death chains.
- Joint convergence to a Poisson surface:

$$\left(C_k^{(T)}(t, s),\ k \in \mathbb{N},\ t \leq 0, s \geq 0 \right) \longrightarrow (N_k(t, s))\,.$$

- Yule process in dimension, birth-and-death chains in time.
- Markov field. Stationary along axis. Joint law by intertwining.

Diffusion limit

Large dimension, small time

- Take centered+scaling limit as $d \to \infty$ and

$$t = -T_0 + u, \quad s = ve^{-T_0}, \quad T_0 \to \infty,\ u \geq 0, v \geq 0.$$

- Large dimension; very small time.
- Imagine observing random transposition chain acting on infinite symmetric group.

Ornstein-Uhlenbeck

Theorem (Johnson-P. '12, Ganguly-P. '14)

Joint convergence to Gaussian field:

$$(2d-1)^{-k/2}\left(2kN_k(-T_0+u, ve^{-T_0}) - \mathbb{E}\left(\cdot\right)\right) \longrightarrow \left(U_k(u,v)\right).$$

- $U_k(\cdot,\cdot)$ *- continuous Gaussian surfaces, independent among* k.
- *Infinite-dimensional O-U surface. Marginally* $N(0,k/2)$.
- *In dimension and time* (U_k) *time-changed stationary O-U:*

$$dU_k(t,\cdot) = -kU_k(t,\cdot)dt + kdW_k(t), \quad t \geq 0.$$

Comparison with Wigner

- Recall $2kN_k \approx \mathrm{tr}\,(T_k(\cdot))$.
- Allows to compute covariances of polynomials linear eigenvalue statistics.
- Same as GOE. A diffusion dynamics on the Gaussian Free Field.

Thank you Jim for all the beautiful math and happy birthday.