Rigidity in Markovian maximal couplings.

Sayan Banerjee

(Joint work with Wilfrid S. Kendall).

University of Warwick

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Maximal Couplings

A coupling of Markov processes $X$ and $Y$ with laws $\mu$ and $\nu$, with coupling time $T$, is called a **Maximal Coupling** if

$$P(T > t) = ||\mu_t - \nu_t||_{TV},$$

for all $t > 0$, where

- $\mu_t$ and $\nu_t$ are distributions of $X_t$ and $Y_t$ respectively.
- $|| \cdot ||_{TV}$ is the total variation distance between measures.
Existence

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- Pitman’s construction simulates the meeting point first and then constructs the forward and backward chains.

- The coupling cheats by looking into the future.
Markovian couplings

A coupling of Markov processes $X$ and $Y$ starting from $x_0$ and $y_0$ is called **Markovian** if

$$(X_{t+s}, Y_{t+s}) \mid F_s$$

is again a coupling of the laws of $X$ and $Y$ starting from $(X_s, Y_s)$. Here $F_s = \sigma\{(X_{s'}, Y_{s'}) : s' \leq s\}$. 
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- The coupling is **not allowed to look into the future**.
When is it possible for two Markov processes to have a Markovian maximal coupling (MMC)?

We investigate this question for diffusions of the form

$$dX_t = b(X_t)dt + dB_t.$$
Known Examples

- *Reflection Coupling* of Brownian motion and Ornstein-Uhlenbeck processes.
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- Kuwada (2009): Brownian motion on a homogeneous Riemannian manifold can be coupled by MMC if and only if the manifold has a reflection structure.
Structure of the MMC

In order to have a MMC, the coupling should satisfy the following:

- There is a deterministic system of mirrors $\{M(t)\}_{t \geq 0}$ which can evolve in time such that, for each $t$, $Y_t$ is obtained by reflecting $X_t$ in $M(t)$.

- Under suitable regularity assumptions, the moving mirror can be parametrized in a smooth way.

- These lead to (implicit) functional equations on the drift, via stochastic calculus.
Theorem
(B’-Kendall)

If there exist \( x_0, y_0 \in \mathbb{R}^d \) and \( r > 0 \) such that there exists a Markovian maximal coupling of the diffusion processes \( X \) and \( Y \) starting from \( x \) and \( y \) for every \( x \in B(x_0, r) \) and \( y \in B(y_0, r) \), then there exist a real scalar \( \lambda \), a skew-symmetric matrix \( T \) and a vector \( c \in \mathbb{R}^d \) such that

\[ b(x) = \lambda x + Tx + c \]

for all \( x \in \mathbb{R}^d \).
Stronger version for one dimension

Theorem
(B’-Kendall)

There exists a Markovian maximal coupling of one dimensional diffusions $X$ and $Y$ starting from $x_0$ and $y_0$ respectively if and only if the drift $b$ is either linear or $b(x) = -b(x_0 + y_0 - x)$ for all $x \in \mathbb{R}$.

Remark: This determines all one dimensional diffusions (with general diffusion coefficient) for which MMC holds, via scale functions.
Conclusion and Remarks

- There exists a complete characterization for time-nonhomogeneous drifts.

- Towards general multidimensional diffusions / diffusions on manifolds, work in progress.

- When MMC does not exist, we can look at efficient couplings (coupling rate of same order as T.V. distance). Some work in this direction has been done for Kolmogorov Diffusions.

Thank You!