The quantile transform of Brownian motion

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Collaborators

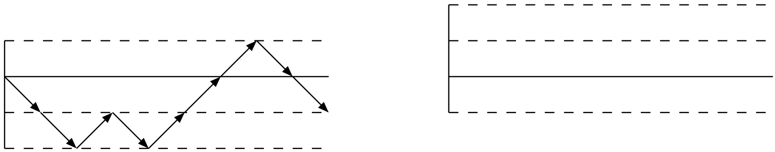
Based on joint work with Sami Assaf (University of Southern California) and Jim Pitman (UC Berkeley).

Part 1

The quantile transform of a walk

Quantile path transformation

The quantile transform reorders the increments of real-valued walks of finite length, based on the value of the walk at the left endpoint of each increment.



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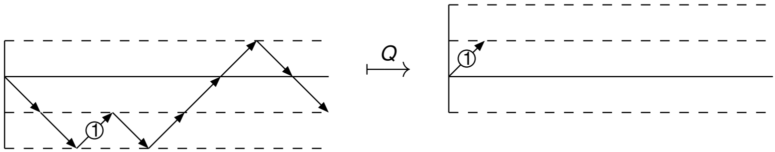

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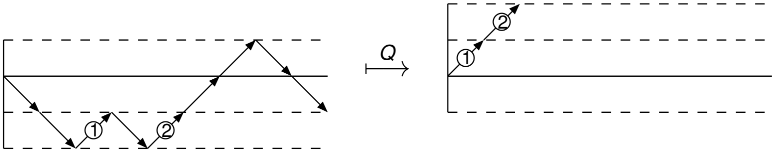

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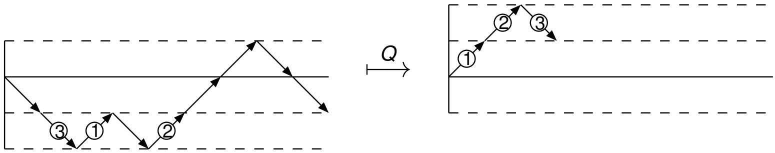

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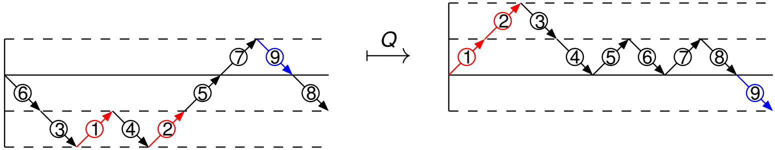


Increments that arise at low points in the walk w are set at the beginning of $Q(w)$; increments that arise at high points of w are set at the end of $Q(w)$.

Previous work on the quantile permutation

In the discrete setting:

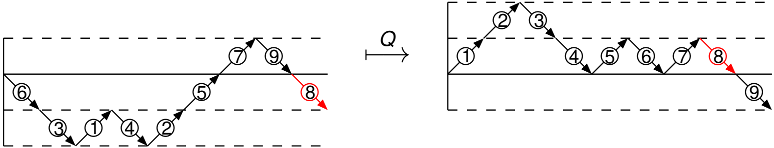
1. Sparre Andersen's Theorem ('53)
2. Spitzer's Combinatorial Lemma ('56)
3. Wendel ('60)
4. Port ('63)
5. Chaumont ('99)

Continuous-time extensions of Wendel's and Port's identities:

1. Dassios ('95, '96, '05)
2. Embrechts, Rogers, and Yor ('95)
3. Bertoin, Chaumont, and Yor ('97)
4. Chaumont ('99, '00)

Notation

$\alpha(w)$ – the time at which the final increment of a walk w appears in $Q(w)$.



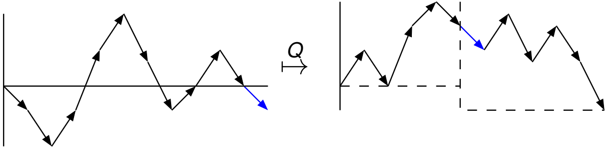
Quantile pair – a walk-index pair (v, k) such that v has finite length n,

$$\begin{aligned} v(j) &\geq 0 && \text{for } j \in [0, k-1], \text{ and} \\ v(j) &> v(n) && \text{for } j \in [k, n-1]. \end{aligned}$$

Results I – general walks

Theorem (AFP, 2013)

For w a real-valued walk, the pair $(Q(w), \alpha(w))$ is a quantile pair.



Corollary (AFP, '13)

Let $(w(j),\ j \in [0,n])$ be a real-valued walk. If $w(n) \geq 0$ then $Q(w)$ is nowhere negative. If $w(n) < 0$ then $Q(w)$ is a first-passage bridge to a negative value.

Results II – simple walks

- Simple walk – a walk of finite length with all increments being ± 1.
- Simple quantile pair – a quantile pair (v, k) in which v is simple.

Theorem (Quantile bijection, AFP, '13)

The map $w \mapsto (Q(w), \alpha(w))$ bijects simple walks with simple quantile pairs.

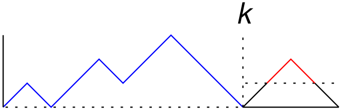
Results III – the bridge case

Q maps simple walk bridges to Dyck paths. Each Dyck path arises in the image with multiplicity equal to the length of its final excursion.

This gives the (previously known) identity

$$\binom{2n}{n} = \sum_{k=1}^{n} 2k\, C_{k-1}\, C_{n-k}$$

where C_j denotes the j^{th} Catalan number.



Part 2

The quantile transform of Brownian motion

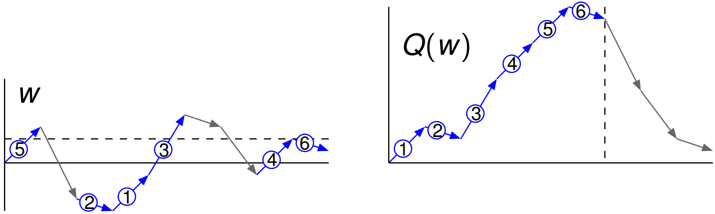
Notation

- Let $(B(t),\ t \in [0, \infty))$ be Brownian motion.
- Let $(\ell(y),\ y \in \mathbb{R})$ denote the (occupation density) local time of B up to time 1 at height y.
- Let $(a(s),\ s \in [0, 1])$ denote the quantile function of occupation measure of B, so

$$\int_0^1 \mathbf{1}\{B(t) \leq a(s)\}dt = s = \int_{-\infty}^{a(s)} \ell(y)dy.$$

Quantile transform and Tanaka's formula

With j fixed, $Q(w)(j)$ is a sum of increments of w that arise at or below some height.



This evokes Tanaka's formula:

$$\int_0^1 \mathbf{1}\{B(s) \leq y\} dB(s) = \frac{1}{2}\ell^y + (y)_+ - (y - B(1))_+.$$

Results I – convergence

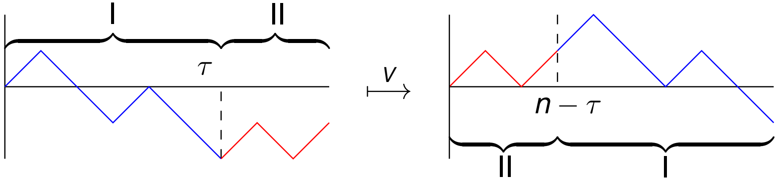
Theorem (AFP, '13)

The quantile transform of a sequence of (suitably rescaled) simple random walks (SRWs) embedded in Brownian motion $(B(t),\ t \in [0,1])$ *converge a.s. uniformly to*

$$Q(B)(t) := \frac{1}{2}\ell^{a(t)} + (a(t))_+ - (a(t) - B(1))_+.$$

Vervaat transform

The Vervaat transform V (Vervaat, '79) splits the increments of a walk w at the first visit to the minimum. It swaps blocks of increments: increments from after the min go at the start of $V(w)$, and those from before the min appear at the end of $V(w)$.



Theorem (AFP, '13)

For S a SRW of finite length, $V(S) \stackrel{d}{=} Q(S)$.

Results II – distribution of the local time profile

The Vervaat transform extends to Brownian motion, and Vervaat ('79) gives a convergence result analogous to ours for the quantile transform.

This leads to the following identity.

Theorem (AFP, '13)

$$Q(B) = \tfrac{1}{2}\ell^{a(t)} + (a(t))_+ - (a(t) - B(1))_+ \stackrel{d}{=} V(B).$$

Jeulin's theorem

Let B^{br} denote a standard Brownian bridge and B^{ex} a standard Brownian excursion.

Theorem (Vervaat, '79)

$V(B^{br}) \stackrel{d}{=} B^{ex}$.

In the Brownian bridge case, our identity between $Q(B)$ and $V(B)$ gives a novel proof of Jeulin's theorem:

Theorem (Jeulin, '85)

For the local time process of B^{br}, we get $\frac{1}{2}\ell^{a(\cdot)} \stackrel{d}{=} B^{ex}(\cdot)$.