Exchangeable graph-valued Markov processes

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Graph-valued Markov process

- $G_N$: graphs with vertex set $\mathbb{N} = \{1, 2, \ldots\}$.
- $G_n$: graphs with vertex set $[n] := \{1, \ldots, n\}$.
- adjacency matrix/array: $G = (G_{ij})_{i,j \geq 1} \in G_N$.
- relabeling: $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ (permutation), $G \mapsto G^\sigma := (G_{\sigma(i)\sigma(j)})_{i,j \geq 1}$.
- restriction: $G_N \rightarrow G_n$, $(G_{ij})_{i,j \geq 1} \mapsto G|_n := (G_{ij})_{1 \leq i,j \leq n}$.

$\Gamma = (\Gamma_t)_{t \geq 0}$ is a Markov process on $G_N$ satisfying

- exchangeability: for all $\sigma: \mathbb{N} \rightarrow \mathbb{N}$, $\Gamma^\sigma := (\Gamma_t^\sigma)_{t \geq 0}$ is a version of $\Gamma$.
- (Markovian) consistency: $\Gamma^n := (\Gamma_t|_n)_{t \geq 0}$ is a Markov chain on $G_n$, for every $n = 1, 2, \ldots$.
An exchangeable graph $\Gamma$ is a weakly exchangeable $\{0, 1\}$-valued array $\Gamma = (\Gamma_{ij})_{i,j \geq 1}$.

Aldous–Hoover theorem: $\Gamma = \mathcal{L} \Gamma^* = (\Gamma^*_{ij})_{i,j \geq 1}$ with

$$
\Gamma^*_{ij} = f(\alpha, \xi_i, \xi_j, \eta_{\{i,j\}}), \quad i,j \geq 1,
$$

where $f(\cdot, b, c, \cdot) = f(\cdot, c, b, \cdot)$ and $\alpha, (\xi_i)_{i \geq 1}, (\eta_{\{i,j\}})_{1 \leq i < j}$ are i.i.d. Uniform$[0,1]$.

(I) overall effect: $\alpha$

(II) vertex effect: $\{\xi_i\}_{i \geq 1}$

(III) edge effect: $\{\eta_{\{i,j\}}\}_{1 \leq i < j}$
Theorem

$\Gamma = (\Gamma_t)_{t \geq 0}$ an exchangeable, consistent Markov process on $\mathcal{G}_N$. Then there are three types of discontinuity:

(I) **global jump**: a positive fraction of all edges changes status;

(II) **single-vertex jump**: a positive fraction of edges incident to a single vertex change, everything else stays the same;

(III) **single-edge flip**: a single edge changes status, everything else stays the same.
Characterization of discontinuities

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**Theorem**

\[ \Gamma = (\Gamma_t)_{t \geq 0} \text{ an exchangeable, consistent Markov process on } \mathcal{G}_N. \] The jump measure decomposes into three parts:

(I) **unique \( \sigma \)-finite measure on \( \{0, 1\} \times \{0, 1\} \)-valued arrays:** random function \( W : [0, 1]^4 \times \{0, 1\} \to \{0, 1\} \) (weakly exchangeable array) so that \( \Gamma_t \mapsto \Gamma_t \) with

\[ \Gamma_t(i, j) = W(\alpha, \xi_i, \xi_j, \eta_{\{i,j\}}, \Gamma_t^-(i, j)), \]

where \( \{\alpha; (\xi_i); (\eta_{\{i,j\}})\} \) are i.i.d. Uniform\([0,1]\).

(II) **unique \( \sigma \)-finite measure on 2 \( \times \) 2 stochastic matrices:** there is a unique \( i = 1, 2, \ldots \) for which \( (\Gamma_t^-(i, 1), \Gamma_t^-(i, 2), \ldots) \) jumps according to a 2 \( \times \) 2 stochastic matrix \( S \).

(III) **unique constants** \( c_{01}, c_{10} \geq 0 \): determine jump rates of each edge.

**Comments:**

- Compare to the Lévy-Itô characterization of exchangeable coalescent processes (binary coagulation, multiple collisions).
  
  (I) **binary coagulation:** two blocks merge, everything else stays the same (“continuous jumps”);
  
  (II) **multiple collisions:** multiple blocks merge simultaneously (“discrete jumps”).

- There is an associated projection of \( \Gamma \) into the space of graph limits.